Chapter 3 Solving Equations and Inequalities

As you saw in the last chapter, when graphing the x-intercepts are very useful in graphing many different functions like the polynomial and rational functions. The information on x-intercepts allows us to do a rough sketch of the functions and so much more. However we also saw we encountered functions whose x-intercepts were elusive to us like $3x^3 + x^2 + 3x + 1 = 0$ and we had to use factor by grouping or in the function $\frac{3x^3-3x-1}{(x+1)^2} = 0$ where we actually did not find the exact intercepts but had to estimate based on our graph. So now we take an algebraic perspective of the functions and see what we can do.

Most of you have been exposed to pre-requisite materials before getting into College Algebra. So we will assume you have some preliminary knowledge solving some linear and non-linear inequalities. If you do not have the pre-requisite knowledge we suggest studying Module 3 in the Developmental and Intermediate Algebra e-text.

3.1 Quadratic Equations

Recall that

An **equation** is a statement asserting the equality of two mathematical objects.
An **inequality** is statement asserting that one mathematical object is larger than or equal to, or less than or equal to another mathematical object.

So we can actually create an equation with equating to different types of functions.
Examples of equations and inequalities

1. $2^x = 3$
2. $2^{x+3} = 2^{4x-1}$
3. $x^2 - 3x = 5(x+1) - 3$
4. $2^x < 3.5$
5. $2^{x+3} \geq 2^{4x-1}$
6. $x^2 - 3x \leq 5(x+1) - 3$

The symbols above can be used as follows:

1 The equation $a = b$ signifies that the mathematical object a is equal to the mathematical object b.
2 The inequality $a < b$ signifies that the mathematical object a is smaller than or less than the mathematical object$b$.
3 The inequality $a > b$ signifies that the mathematical object a is bigger than or greater than the mathematical object b.
4 The inequality $a \leq b$ signifies that the mathematical object a is smaller than or equal to, or less than or equal to, the mathematical object$b$.
5 The inequality $a \geq b$ signifies that the mathematical object a is bigger than or equal to, or greater than or equal to, the mathematical object b.

The symbols above are our shorthand way of writing comparisons between two mathematical objects. In this way, you can see how mathematics is a symbolic language.

All real number solutions to an equation or inequalities in one variable can be represented on a real number line. All equations in one variable can be written in such a way that all the non-zero terms are on the left side and zero on the right side. This makes solutions to these equations candidates for the x-intercepts of functions $f(x) =$ left hand side of the equation in one variable. All real number solutions to an equation or inequality in two variables can be represented as points or regions in a 2-dimensional Cartesian coordinate system. When we equate two non-zero functions like $x^2 - 3x = 3x - 1$ we are trying to see at what x values can the two function intersect each other.

Any number(s) when substituted for the variable(s) in the original equation or inequality that results in a true statement is called a **solution** to that equation or inequality.
The process in which we use mathematical properties of equality or inequality respectively to isolate the variable by itself is called **solving** the equation or inequality.

Solving an equation or an inequality is like untying a knot or undoing what was done to the variable to get it into its current state.

An equation that is true for all values of the variable is called an identity.

The equation $x \times x = x^2$ is an identity. Both sides of the equal sign yield the same number for any real number value of x.

An equation in which you end up with a false statement for all values of the variable is said to have "**No Solution**".

Certain tools used in the process of isolating the variable can sometimes lead us to a value of the variable that makes the equation false. Such a solution is called an **extraneous** solution and we will need to watch out for these "false" solutions whenever we use those tools.

Before we attempt to solve polynomial equations let us review complex numbers.
For a long time people believed that the set of real numbers accounted for all the numbers that there are. However in 1545, Girolamo Cardano and others made progress on solving cubic equations such as $x^3 - 15x = 4$. Their technique gave a solution to this problem that involved $\sqrt{-121}$. The value of this square root cannot be any real number since any real number when squared will be positive. However their solution method simplified to a final result of $x = 4$ which is easily seen to make the above equation true. This lead to a deeper study of square roots of negative numbers. Today these kinds of numbers are absolutely essential in higher mathematics, engineering and physics. A simpler problem that shows the deficiency of the real numbers is to try to find a real number x such that $x^2 = -1$. Play with this for a while to convince yourself that no real number when multiplied by itself will produce the result negative one!

So a new kind of number system evolved where $i = \sqrt{-1}$ is designated to represent the unit imaginary number a solution to the equation $x^2 = -1$. So even though it is hard to imagine what i is, we know that its square is -1 or that $i \times i = -1$. With this definition of i we expand the set of real numbers to the set of **complex numbers.**

Set of complex numbers is a collection of all numbers of the form $a + bi$, where a, b are real numbers and are called the **real part** and **imaginary part** respectively. Another way to represent this sentence in mathematical notation is $C = \{a + bi \mid a \text{ and } b \text{ are any real numbers and } i^2 = -1\}$.

Note: *The set of all real numbers is a subset of the set of complex numbers because every real number can be written as* $a + 0i$, *e.g.,* $2.4 = 2.4 + 0i$ *in set notation it will look like* $R \subset C$.

We can do arithmetic with complex numbers using properties of arithmetic and using the fact you can add or subtract like terms.

Practice Problems

1. Simply the following and write all your answers in the $a + bi$ form where a, and b are real numbers.
   a) $(3 + 5i) + (2 - 3i)$
      Adding like terms we get $(3 + 5i) + (2 - 3i) = (3 + 2) + (5 - 3)i = 5 + 2i$
   b) $(3 + 5i) - (2 - 3i)$
      $(3 + 5i) - (2 - 3i) = (3 - 2) + (5 - (-3))i = 1 + 8i$
   c) $(3 + 5i)(2 - 3i)$
      Using distributive property and adding like terms we get
      $(3 + 5i)(2 - 3i) = 3(2) + 3(-3i) + 5i(2) + 5i(-3i)$
      $= 6 - 9i + 10i - 15i^2$ (recall $i^2 = -1$) giving us
      $= 6 + i + 15$
      $= 21 + i$
   d) $(2 + 3i)(2 - 3i)$
      $= 4 - 6i + 6i - 9i^2$
      $= 4 + 9$
      $= 13$

Such pairs of numbers $2 + 3i$ and $2 - 3i$ are called conjugates of each other.

Conjugates: The pair of complex numbers $a + bi$ and $a - bi$ are called conjugates of each and they multiply to give us a real number.
$(a + bi)(a - bi) = a^2 + b^2$

e) $\frac{3+5i}{2-3i}$

Since we need to write the division in $a+bi$ form we need multiply numerator and denominator by the conjugate of $2-3i$ which is $2+3i$

$$\frac{3+5i}{2-3i}=\frac{(3+5i)(2+3i)}{(2-3i)(2+3i)}=\frac{6+9i+10i+15i^2}{4+6i-6i-9i^2}$$
$$=\frac{6+19i-15}{4+9}=\frac{-9+19i}{13}=-\frac{9}{13}+\frac{19}{13}i$$

f) Simplify the powers of i below. Do you notice any patterns

$i^1=i$	$i^2=-1$	$i^3=-i$	$i^4=1$
$i^5=i$	$i^6=-1$	$i^7=-i$	$i^8=1$
$i^9=i$	$i^{10}=-1$	$i^{11}=-i$	$i^{12}=1$

Note that every 4 the powers of i start to repeat. So we have in general

- $i^n=1$ if and only if n is a multiple of 4.
- $i^n=-1$ if and only if n is a multiple of 2 but not 4.
- $i^n=i$ if and only if n divided by 4 leaves a remainder of 1.
- $i^n=-$i if and only if n divided by 4 leaves a remainder of 3.

g) i^{401}

Since 401 divided by 4 leaves a remainder of 1 we get $i^{401}=i$

h) $\sqrt{-81}$

$\sqrt{-81}=9i$

i) $\sqrt{-8}$

$\sqrt{-8}=\sqrt{8}\,i=2\sqrt{2}\,i$

2. Solve the following equations for the given variable. If there is more than one solution, separate them with commas.

a. $4\,x^2+15\,x+9=28$

Step 1:Simplify and gather all terms on one side and have **<u>zero</u>** on the other side.

Step 2:Pull out greatest common factor if necessary.

Step 3: Factor

Step 4. Solve using zero property of equality.

$4\,x^2+15\,x+9-28=28-28$

$4\,x^2+15\,x-19=0$

$(x-1)(4x+19)=0$

$x-1=0,\quad or\ \ 4x+19=0.$

$x=1,\ \ or\ \ 4x=-19$

Thus the solutions are $x=1,\ \ or\ \ x=-\frac{19}{4}.$

b. $2x^2 - 10x - 10 = 0$

Completing the squares is easiest with the coefficient of the x^2 as 1. Rewriting the equation so that the coefficient of the square term is 1 and the constant term is on the right hand side we get

$2x^2 - 10x = 10$ (divide both sides by 2)

$$x^2 - 5x = 5$$

$$x^2 - 5x + \left(-\frac{5}{2}\right)^2 = 5 + \left(-\frac{5}{2}\right)^2$$

$$\left(x - \frac{5}{2}\right)^2 = 5 + \frac{25}{4}$$

$$x - \frac{5}{2} = \pm\sqrt{\frac{20}{4} + \frac{25}{4}}$$

$$x = \frac{5}{2} \pm \sqrt{\frac{45}{4}}$$

$$x = \frac{5}{2} \pm \frac{\sqrt{45}}{\sqrt{4}} = \frac{5}{2} \pm \frac{3\sqrt{5}}{2}$$

c. $2x^2 - 10x + 15 = 0$

Rewriting the equation so that the coefficient of the square term is 1 and the constant term is on the right hand side we get

$2x^2 - 10x = -15$ (divide both sides by 2)

$$x^2 - 5x = -\frac{15}{2}$$

$$x^2 - 5x + \left(-\frac{5}{2}\right)^2 = -\frac{15}{2} + \left(-\frac{5}{2}\right)^2$$

$$\left(x - \frac{5}{2}\right)^2 = -\frac{15}{2} + \frac{25}{4}$$

$$x - \frac{5}{2} = \pm\sqrt{\frac{-30}{4} + \frac{25}{4}}$$

$$x = \frac{5}{2} \pm \sqrt{\frac{-5}{4}}$$

$$x = \frac{5}{2} \pm \frac{\sqrt{5}}{2} i$$

In general if we complete the squares on a quadratic equation in one variable we get a generic formula that can now be used avoiding having to complete squares regularly.

If we apply completing the squares to a generic quadratic formula we get the quadratic formula

$ax^2 + bx + c = 0$

$ax^2 + bx = -c$

(Get the variable terms alone on one side.)

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

(Make the x^2 term have coefficient of 1.)

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

(Make the left side a perfect square by adding the square of half of the x-coefficient.)

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{4a(c)}{4a(a)} + \frac{b^2}{4a^2}$$

(Get a common denominator on the right.)

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

(Take square root of both sides.)

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic Formula: The solutions to the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0, b, c$ are real numbers are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note: *This means that the solutions to the quadratic equation $ax^2 + bx + c = 0$ are given by*

$$x = \frac{-b+\sqrt{b^2-4ac}}{2a}, \text{ or } x = \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

The term inside the root $b^2 - 4ac$ is called the discriminant as it controls what kind of solutions we get.

In fact:

1. If the discriminant $b^2 - 4ac > 0$, then the quadratic equation has two distinct real solutions.

2. If the discriminant $b^2 - 4ac = 0$, then the quadratic equation has one real solution.
3. If the discriminant $b^2 - 4ac < 0$, then the quadratic equation has two distinct complex solutions.

Practice Examples

1. $3x^2 - x + 3 = 0$
   Note that $a = 3, b = -1, c = 3$
   Using the quadratic formula we get
   $$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(3)}}{2(3)}$$
   $$x = \frac{1 \pm \sqrt{1-36}}{6}$$
   $$x = \frac{1 \pm \sqrt{-35}}{6} = \frac{1 \pm \sqrt{35}i}{6}$$
   $$x = \frac{1}{6} + \frac{\sqrt{35}}{6}i \text{ or } x = \frac{1}{6} - \frac{\sqrt{35}}{6}i$$
   Note that here $b^2 - 4ac < 0$.

2. $4x^2 - 20x + 25 = 0$
   Note that $a = 4, b = -20, c = 25$
   Using the quadratic formula we get
   $$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(4)(25)}}{2(4)}$$
   $$= \frac{20 \pm \sqrt{400-400}}{8} = \frac{20 \pm 0}{8} = \frac{20}{8} = \frac{5}{2}$$
   Or by factoring:
   $4x^2 - 20x + 25 = 0$
   $(2x - 5)^2 = 0$
   $2x - 5 = 0$
   $2x = 5$
   $x = \frac{5}{2}$

Note: Another way to say that $4x^2 - 20x + 25 = 0$ has zero of $x = \frac{5}{2}$ repeated twice is to say that $4x^2 - 20x + 25 = 0$ has a zero of $x = \frac{5}{2}$ with a multiplicity of 2.

3. $3x^4 + 14x^2 + 8 = 0$
   Factoring we get $(3x^2 + 2)(x^2 + 4) = 0$
   $x^2 = 4$ or $3x^2 = -2$
   $x = \pm\sqrt{4} = \pm 2$ or $x = \pm\sqrt{\frac{2}{3}}i = \pm\frac{\sqrt{2}}{\sqrt{3}}i = \pm\frac{\sqrt{2}\sqrt{3}}{\sqrt{3}\sqrt{3}}i$
   $x = 2, -2, \frac{\sqrt{6}}{3}i, -\frac{\sqrt{6}}{3}i$

Remember the graph of a quadratic function in one variable represents the parabola. So finding the solutions to a quadratic equations finds the x-intercepts of the parabola if they exist.

Cover Sheet for Sec 3.1 Name: ___________________________

1. What is a complex number?

2. What is the quadratic formula? Explain how it works.

3. What is a discriminant?

4. How would you find the maximum or minimum of a parabola?

5. Can you use your knowledge of solving quadratic equations, exponential and logarithmic functions to solve the following problems?
    a. $2^{2x} - 2^x - 2 = 0$
    b. $x^4 - 3x^2 - 4 = 0$

Video log 3.1

Simplify the following and write your answer in standard $a + bi$ form.

1. $\sqrt{-72}$
2. $\frac{\sqrt{-66}}{\sqrt{-6}}$
3. $\sqrt{-121}\sqrt{144}$
4. $(3 + 5i) + (-5 + 6i)$
5. $\frac{4-2i}{-2-5i}$
6. i^{35}

Solve the following equations for the given variable. If there is more than one solution, separate them with commas.

7. $(5y + 4)(2y - 3) = 0$
8. $u^2 - 10u + 21 = 0$
9. $5w^2 = 17w - 6$
10. $x^2 - 10x + 10 = 0$
    (by completing the square)
    Form:
    - $(x +$ ____ $)^2 =$ ______
    - $(x -$ ____ $)^2 =$ ______

    Solution

    $x =$ ________
11. $2x^2 + 5x - 1 = 0$
12. $2x^2 - 3x + 6 = 0$

Find the discriminant and determine the number of real solutions of the quadratic equation. Then find the actual solutions

13. $4x^2 - 12x + 9 = 0$

Discriminant:

Number of solutions:

Actual solutions:

14. $-2x^2 - 6x + 8 = 0$

Discriminant:

Number of solutions:

Actual solutions:

15. $2x^2 - 5x + 8 = 0$

Discriminant:

Number of solutions:

Actual solutions:

Determine all the solutions to the equations below. If there is more than one solution, separate them with commas.

16. $x^4 - 5x^2 - 6 = 0$

17. $x^{4/3} - 5x^{2/3} - 6 = 0$

18. $y^6 - 5y^3 - 6 = 0$

19. $\frac{1}{x^2} - \frac{37}{x} + 36 = 0$

20. $(x-1)^2 - 37(x-1) - 36 = 0$

21. $u^4 + 2u^2 + 1 = 0$

22. $2x(x-1)^3(2x+3)^4(5-x)^2(x^2+4)=0$

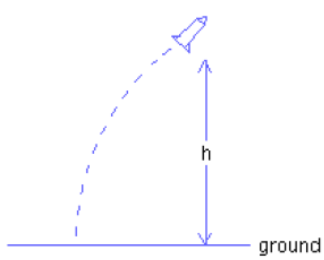
Solve the following problems. If there is no solution, please state so.

23. The length of a rectangle is 5 yd less than twice the width, and the area of the rectangle is 33 yd^2. Find the dimensions of the rectangle.

24. A rocket model is launched with an initial velocity of 235 ft/s. The rocket's height h (in feet) after t seconds is given by the following.

$$h = 235t - 16t^2$$

Find all the values of t for which the rocket's height is 151 feet. Round your answers to the nearest hundredth. If there is more than one answer, use or to separate them.



25. The cost C in (dollars) of manufacturing x wheels at Ravi's Bicycle Supply is given by the function $C(x) = 0.5x^2 - 170x + 25{,}850$. What is the minimum cost of manufacturing wheels? Do not round your answer.

REVIEW for Parabolas or Quadratic Functions.
For the following quadratic function (parabolas), determine what information of the function is hidden in the equations below. Axis of symmetry, vertex, x-intercepts, y-intercepts, maximum's, minimum's. In your own words describe what the graph of a quadratic function looks like and the information you can determine by solving or manipulating the equations.

26. $f(x) = -2x^2 + 16x - 34$

Complete the squares

Solving $-2x^2 + 16x - 34 = 0$

Finding $f(0)$

Find $f(4)$

3.2 Polynomial Equations of Degree Three or Higher

We saw at the of Chapter 2 how to find possible rational zeros.

In mathematics we cannot just state what we observe in particular cases as the truth for all cases. So we make our observations and propose a statement that is not self-evident but can be proved by a chain of mathematical reasoning to prove our statement. Such statements that can be proved are called theorem.

Rational Zeros Theorem:

Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n,$ where $a_0,\ a_1,\ a_2,\ a_{3,} \ldots a_n \neq 0$ are all integers and $n \geq 0$ is a whole number be a polynomial function with degree n. The rational number in simplest form $x = \frac{p}{q}$ is a rational zero of the polynomial function $f(x)$ if $p =$a factor of the constant term in the polynomial, and $q =$ a factor of the leading coefficient.

Proof: To prove this statement lets start with the function what is given

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n$, and that $x = \frac{p}{q}$ is a rational zero of this polynomial.

That means $f\left(\frac{p}{q}\right) = 0$ or that $a_0 + a_1\left(\frac{p}{q}\right) + a_2\left(\frac{p}{q}\right)^2 + a_3\left(\frac{p}{q}\right)^3 + \cdots + a_n\left(\frac{p}{q}\right)^n = 0$.

Multiply both sides q^n. We get

$a_0q^n + a_1pq^{n-1} + a_2p^2q^{n-2} + a_3p^3q^{n-3} + \cdots + a_np^n = 0.$

$a_1pq^{n-1} + a_2p^2q^{n-2} + a_3p^3q^{n-3} + \cdots + a_np^n = -a_0q^n.$

$p(a_1q^{n-1} + a_2pq^{n-2} + a_3p^2q^{n-3} + \cdots + a_np^{n-1}) = -a_0q^n$

This means that the left hand side is divisible by p and therefore a_0q^n is divisible by p. Since p,q are in simplest form they do not share any common factors giving is that p must be factor of a_0

So now we know that the numerator of the rational zero must be a factor of the constant term.

Rewriting the $a_0q^n + a_1pq^{n-1} + a_2p^2q^{n-2} + a_3p^3q^{n-3} + \cdots + a_np^n = 0$ as

$a_0q^n + a_1pq^{n-1} + a_2p^2q^{n-2} + a_3p^3q^{n-3} + \cdots + a_{n-1}p^{n-1}q = -a_np^n$

$q(a_0q^{n-1} + a_1pq^{n-2} + a_2p^2q^{n-3} + a_3p^3q^{n-3} + \cdots + a_{n-1}p^{n-1}) = -a_np^n$

That means the left hand side is divisible by q, therefore a_np^n is divisible by q. Since p,q are in simplest form they do not share any common factors giving is that q must be factor of a_n.

This proves our theorem.

Recall that when doing long division of numbers of polynomials we can use the division algorithm to write the dividend in terms of the divisor.

Division Algorithm: Let $f(x)$ be a polynomial with real coefficients. Let $f(x) \div (x-a)$ for a real number a have a remainder of $R(x)$ and quotient of $Q(x)$. Then $f(x) = Q(x)(x-a) + R(x)$

We can see by this algorithm then that $f(a) = Q(a)(a-a) + R(a) = R(a)$.

Remainder Theorem: Let $f(x)$ be a polynomial with real coefficients. Then $x = a$ is a zero of $f(x)$ if and only if $f(a) = R(a) = 0$. In other words the remainder can of $f(x) \div (x-a)$ can be found by evaluating the function at $x = a$.

Factor Theorem: Let $f(x)$ be a polynomial with real coefficients. Then $x - a$ is a factor of $f(x)$ if and only if $f(a) = 0$.

Again you would have to prove these statements. See if you can generate the proofs on your own.

We have also seen now that not all polynomials have real zeros. If they do have real zeros, they may not be rational zeros. For example, $x^2 - 2 = 0$ gives us $x = \sqrt{2}$ or $-\sqrt{2}$ as its zeros neither are rational numbers. In fact, we have seen that sometimes we get non-real zeros as well. But if we knew that the polynomial has real coefficients then the non-real zeros must come in conjugate pairs otherwise we would have coefficients that are non-real.

Fundamental Theorem of Algebra: Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n,$ be a polynomial with real coefficients with $a_n \neq 0$ and degree $n \geq 1$. Then $f(x)$ can be written as a product of linear and quadratic polynomials where the quadratic polynomials havenon-real zeros or are irreducible polynomials over the set of real numbers.

The proof of this will be given later.
Using this theorem and the other knowledge accumulated so far we can make a lot of conclusions about a polynomial with real coefficients.

Practice Problems

1. Suppose $f(x)$ is a polynomial of degree 11 whose coefficients are real numbers. Also, suppose that $f(x)$ has the following zeros: $3, 5i, -2-4i$.

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a) Find another zero of $f(x)$.

Clearly since the polynomial has real coefficient all non-real zeros must occur in conjugate pairs.
So if $5i$ is a zero then automatically $-5i$ is a zero.
Similary if $-2-4i$ is a zero, $-2+4i$ must also be a zero. In addition that means
$(x-5i)(x+5i)\big(x-(-2-4i)\big)(x-(-2+4i))$ are factors of the polynomial
Use distributive property to multiply and we get
$(x-5i)(x+5i)$
$= x^2+5xi-5xi-25i^2$
$= x^2+25$
And
$\big(x-(-2-4i)\big)\big(x-(-2+4i)\big)$
$= (x+2+4i)(x+2-4i)$
$= x^2+2x-4xi+2x+4-8i+4xi+8i-16i^2$
$= x^2+4x+4+16$
$= x^2+4x+20$
So $(x^2+25)(\,x^2+4x+20)$

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b) What is the maximum number of real zeros that $f(x)$ can have?

We know now from part a) solution that there are at least 4 non-real zeros. We also know that the degree of the polynomial is 11. That leaves us with $11-4=7$ additional zeros to account for and they could all be real zeros. So the maximum number of real zeros are 7.

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c) What is the maximum number of non-real zeros that $f(x)$ can have?
Since all non-real zeros occur in conjugate pairs there are always even number non-real zeros. Since our polynomial has one real zero listed and the degree of the polynomial is 11 we can have a maximum of $11-1=10$non-real zeros.

d) If the leading coefficient of the polynomial was -8, what can the polynomial look like? Find a polynomial expression that has all the properties mentioned in parts a)-d)
One example of a polynomial could be

$$f(x) = -3(x-3)^7(x^2+25)(\,x^2+4x+20)$$

This polynomial has all the required characteristics. There could be many other versions like

$$f(x) = -3(x-3)^3(x^2+25)^3(\,x^2+4x+20)$$

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Synthetic Division

Long division of polynomials is similar to the whole number long division as we saw when finding oblique asymptotes. See example below for $6x^2 + 7x + 4 \div \mathrm{x} + 2$. As you can see in long division since the first term subtracts out the second term is the one that is useful and so when dividing a polynomial with terms like $x - a$ we can use something called synthetic division. You use a from the divisor $x - a$ and then instead of subtracting we add as shown below.

Step 1: The coefficients of the dividend polynomial go in the boxes in the top row which for us would be the 6,7,6.

Step 2: The number $a = -2$ goes in the second row to the right most.

Step 3: Then bring the coefficient of x^2 term down as is in the last row.

Step 4: Multiply the first number in the bottom row and put the result in the second row under 7, add the two terms in that column and put that resulting number in the last row in that column, multiply that number by -2 and put that number in the second row last column and continue the process. The last number in the last row is the remainder. The remaining numbers are the coefficient of the quotient which is a polynomial of one degree lower than the dividend.

As you can see below, synthetic division helps and can be used when finding zeros.

Long Division

$$
\begin{array}{r|l}
 & \quad 6x - 5 \\
(x+2) & \quad 6x^2 + 7x + 4 \\
 & -(6x^2 + 12x) \\
\hline
 & \qquad -5x + 4 \\
 & \qquad -(5x - 10) \\
\hline
 & \qquad\qquad 14
\end{array}
$$

Quotient $6x - 5$

Remainder is 14

Synthetic Division

	6	7	4
-2		-12	10
	6	-5	14

Quotient $6x - 5$

Remainder is 14

So now we can use our theory to actually find all the zeros.
Practice Example

1. Find all the zeros of $3x^4 + 2x^3 - 4x^2 - 29x + 10 = 0$
   If the polynomial has rational zeros the
   Possible numerators could be all factors of 10 : 1,2,5,10
   Possible denominators could be all factors of 3: 1,3
   Possible rational zeros could be $\pm 1, \pm\frac{1}{3}, \pm 2, \pm\frac{2}{3}, \pm 5, \pm\frac{5}{3}, \pm 10, \pm\frac{10}{3}$,
   We can now use synthetic division to find the zero.

	3	2	-4	-29	10
-2		-6	8	-8	74
	3	-4	4	-37	84

	3	2	-4	-29	10
2		6	16	24	-10
	3	8	12	-5	0

$x = 2$ is a zero so we can work with the polynomial $3x^3 + 8x^2 + 12x - 5$ which is our quotient

	3	8	12	-5
$\frac{1}{3}$		1	3	5
	3	9	15	0

$x = \frac{1}{3}$ is a zero also and now the quotient is
$3x^3 + 9x + 15 = 0$ using our quadratic formula we get the remaining zeros to be
$x = \frac{-9 \pm \sqrt{81 - 4(3)(15)}}{6}$ after simplifying you will get $x = -\frac{3}{2} + \frac{\sqrt{11}}{2}i, -\frac{3}{2} - \frac{\sqrt{11}}{2}i$
So all our zeros of $3x^4 + 2x^3 - 4x^2 - 29x + 10 = 0$ are
$x = \frac{1}{3}, 2, \ \frac{3}{2} + \frac{\sqrt{11}}{2}i, \ -\frac{3}{2} - \frac{\sqrt{11}}{2}i$

Cover Sheet for Sec 3.2 **Name: ____**

1. What can you say about non-real zeros if the coefficients can be non-real also?

2. What is the maximum number of real zeros a polynomial of odd degree have?

3. What is the maximum number of real zeros a polynomial of even degree have?

4. If you knew you had two non-real zeros, then what is the maximum number of real zeros a polynomial of odd degree have?

5. If you knew you had two non-real zeros, then what is the maximum number of real zeros a polynomial of even degree have?

Video Log 3.2

1. Suppose $R(x)$ is a polynomial of degree 13 whose coefficients are real numbers. Also, suppose that $R(x)$ has the following zeros: $7, -8, 5i, -2-4i$.	
e) Find another zero of $R(x)$.	f) What is the maximum number of real zeros that $R(x)$ can have?
g) What is the maximum number of nonreal zeros that $R(x)$ can have?	h) If the leading coefficient of the polynomial was -3, what can the polynomial look like? Find a polynomial expression that has all the properties mentioned in parts a)-d)
2. Find a polynomial $f(x)$ of degree 4 that has the following zeros $-2, 1, -6, 0$. Leave your answer in factored form.	

3. Perform the following division. If possible use synthetic division.	
a) $(6x^2 + 37x + 39) \div (x + 5)$	b) $(3x^2 - x^3 + 6x - 8) \div (x - 4)$ (write you answer in the Quotient $+ \frac{Remainder}{x-4}$
c) $(24x^3 + 4x^2 + 14x + 3) \div (6x - 2)$	
4. Use the rational zeros theorem to list all possible zeros of the following . Be sure that no value in your list appears more than once.	
a) $g(x) = -5x^4 - x^3 - 3x^2 - 3x + 3$	b) $g(x) = -6x^4 - x^3 - 3x^2 - 3x + 9$

5. Use Descartes's Rule of Signs to determine the possible numbers of positive and negative real zeros. $$g(x) = -4x^3 + 5x^2 - 6x - 5$$ If there is more than one possibility, sepreate them with commas. Possible number of positive real zeros: Possible number of negative real zeros:	6. The function below has at least one rational zero. Use this fact to find all zeros of the function. $$g(x) = 4x^3 + 12x^2 - x - 3$$ If there is more than one zero, separate them with commas. Write exact values, not decimal approximations.
7. The function below has at least one rational zero. Use this fact to find all zeros of the function. $$g(x) = 7x^4 + 20x^3 + 10x^2 - 5x - 2$$ If there is more than one zero, separate them with commas. Write exact values, not decimal approximations.	8. For the polynomial below -2 is a zero. $h(x) = x^3 + 8x^2 + 30x + 36$. Express $h(x)$ as a product of linear factors.

9. The function below has at least one rational zero. Use this fact to find all zeros of the function.

$$h(x) = 5x^4 - 29x^3 - 40x^2 - 13x - 7$$

10. Find all the other zeros of
$P(x) = x^4 - 7x^3 + 22x^2 - 6x - 36$ given that $3-3i$ is a zero.

3.3 Exponential and Logarithmic Equations

To undo a function we can bring inverse functions to help us when they exist.

For example exponential and logarithmic equations can be solved using inverse functions since they are one-to-one functions.

Practice Example

1. Find all real solutions to the equations and inequalities below.

<table>
<tr><th>Equations</th><th>Inequalities</th></tr>
<tr><td>

a) $2^x = 3$

Our x is in the exponent and we know that since logarithmic function base ten is a one to one the value of $\log(2^x)$ must equal $log(3)$ and using properties of logarithms we have

$\log(2^x) = \log(3)$

$\text{x}\log(2) = \log(3)$ and therefore

$$x = \frac{log3}{log2}$$

This is the exact solution and if you use your calculators we will get an approximate solution.

So we have

$$x = \frac{log3}{log2} \approx 1.585$$

Graphically that means the two functions $y = 2^x$ and $y = 3$ intersect at $x = \frac{log3}{log2} \approx 1.585.$

Or that the x-intercept of the function $f(x) = 2^x - 3$ is at

$$x = \frac{log3}{log2} \approx 1.585$$

</td><td rowspan="2">

a) $2^x < 3$

We solve the equation

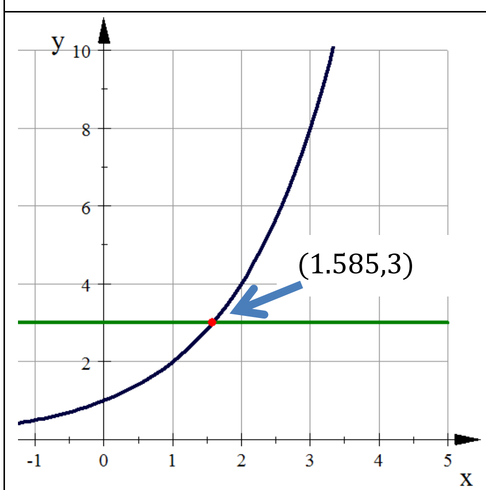
$$2^x = 3$$

As done to the left and then we know we need to have

$$x < \frac{log3}{log2} \approx 1.585$$

Solution: $\left(-\infty, \frac{log3}{log2}\right)$

</td></tr>
<tr><td>



</td></tr>
</table>

b) Half-life of penicillin (a form of antibiotic) is about 35 minutes for an adult with normal renal function (or kidneys). Initial dose for an adult is between 250-3000mg 2 to 4 times a day to treat different kinds of bacterial infections. If an adult was accidentally was given an over does of $6000mg$ in one dose. How long will it take for the does to come down to 200mg assuming that the adult drank a lot of fluids to flush the drug out of their system?

Our equation would have to look like

$$6000\left(\frac{1}{2}\right)^{\frac{t}{35}} = 200$$

To undo here we first divide both sides by 6000 and then we can use the logarithmic function.

$\left(\frac{1}{2}\right)^{\frac{t}{35}} = \frac{200}{6000}$ or $ln\left(\frac{1}{2}\right)^{\frac{t}{35}} = ln\left(\frac{200}{6000}\right)$

$$\frac{t}{35}\ln\left(\frac{1}{2}\right) = ln\left(\frac{200}{6000}\right)$$

$$t = \frac{35ln\left(\frac{200}{6000}\right)}{\ln\left(\frac{1}{2}\right)} \approx 171.74minutes \approx 2\ hours51.6minutes$$

So assuming you drink a lot of fluids to flush things out of the body it will take just a little under 3 hours.

c) $2^{x-5} = 4^{5-x}$

Here we could use logarithmic functions but we know that exponential functions are one-to-one also and that $4 = 2^2$ so we get

$2^{x-5} = (2^2)^{5-x}$

$2^{x-5} = 2^{10-2x}$

That means the exponents must be the same.

$x - 5 = 10 - 2x$

$3x = 15$

Solution: $x = 5$

d) $2^{x-5} = 3^{4-x}$

Here we have to use logarithmic functions.

$\ln(2^{x-5}) = ln(3^{4-x})$

$(x - 5)ln2 = (4 - x)ln3$

Since $ln2$ and $ln3$ are just numbers we treat them like it and solve the equation above as a linear equation can be solved.

$xln2 - 5ln2 = 4ln3 - xln3$

$xln2 + xln3 = 4ln3 + 5ln2$

$x(ln2 + ln3) = 4ln3 + 5ln2$

$$x = \frac{4ln3 + 5ln2}{ln2 + ln3}$$

This is the exact solution to the equation and approximate solution can be found by using your calculators which would give you

$x = \frac{4ln3+5ln2}{ln2+ln3} \approx 4.386$

e) Amy invested \$4000 at 5% interest compounded quarterly. How many years will she will have to wait for her money to grow to $\$5000$. $4000\left(1+\frac{0.05}{4}\right)^{4t} = 5000$ solve this equation for t we use logarithmic function. $\left(1+\frac{0.05}{4}\right)^{4t} = \frac{5000}{4000}$ $4tln\left(1+\frac{0.05}{4}\right) = ln\left(\frac{5}{4}\right)$ $t = \frac{ln\left(\frac{5}{4}\right)}{4ln\left(1+\frac{0.05}{4}\right)} \approx 4.49$ She would have to wait about 4 years and approximately 6 months.
f) $\log(x-1) = 5$ Here we would need to use exponential functions or use the fact that exponential functions are inverse functions of logarithmic functions to give us $x - 1 = 10^5$ $x = 100001$ Check $\log(100001 - 1) = \log(100000) = 5$ It is important to check the answer since the domain of the logarithmic function is restricted.
g) $\log_3(x-1) = -2$ Here we would need to use exponential functions or use the fact that exponential functions are inverse functions of logarithmic functions to give us $x - 1 = 3^{-2}$ $x = 1 + \frac{1}{9} = \frac{10}{9}$ Check your answer please….
h) $\log_3(x-1) = \log_3(5x-7)$ Since logarithmic function is one-to-one we get $x - 1 = 5x - 7$ Or $-4x = -6$ $x = \frac{-6}{-4} = \frac{3}{2}$ Check your answer please….

i) $\log_3(x-1) - \log_3(5x-7) = 2$ We can use properties of logarithms to solve this problem $$\log_3\left(\frac{x-1}{5x-7}\right) = 2$$ Now using the fact that logarithmic function is the inverse function of the exponential function we get $$\frac{x-1}{5x-7} = 3^2$$ $$x-1 = 9(5x-7)$$ $$x-1 = 45x-63$$ $$62 = 44x$$ $$x = \frac{62}{44} = \frac{31}{22}$$ We need to check this solutions so go ahead do that
j) $log2 + \log(x-1) = \log(3x-1)$ We can use properties of logarithms to solve this problem $$\log\big(2(x-1)\big) = \log(3x-1)$$ Using the fact that log functions are one-to-one we get $$2x-2 = 3x-1$$ $$x = -1$$ Note that the domain of $\log(x-1)$ is $x > 1$ so we have an extraneous solution There is no solution to this problem.

So we now know how to solve logarithmic and exponential equations.

Cover Sheet for Sec 3.3 Name: ________________________________

1. What is the mathematical principle that allows you to solve logarithmic and exponential equations?

2. List all the properties of inverse functions that helped you solve logarithmic and exponential equations?

3. List all the properties of logarithmic functions that helped you solve logarithmic or exponential equations?

Video Log 3.3

Find all the solutions to the following equations.
1. $3x - 5 = 2 - 4x$
2. $x(x-3)(2x-3) = 0$
3. $\frac{x+1}{x-3} = 4$
4. $log_3(x+1) = 2$
5. $log_3(x+1) - log_3(x) = 2$
6. $log_3(2x-1) + log_3(x+1) = 2$

7. $log_4(-1-2x)=-1$
8. $4+\log(2x-1)=5$
9. $log_5(x-3)=1-log_5(x-7)$
10. $\ln(x+4)-\ln 18=ln5$
11. $125=25^{-x-2}$
12. $2^{x^2-61x}=64^{3-9x}$

13. $15^{-8y} = 6$
(Round your answer to the nearest hundredth. Do not round any intermediate computations.)

14. $e^{-8u} = 6$
(Round your answer to the nearest hundredth. Do not round any intermediate computations.)

15. $17^{-x-3} = 16^{-8x}$
(Write the exact answer using base-10 logarithms)

16. $3^{x-1} = 5^{2x-1}$
(Write the exact answer using natural logarithms)

17. $200e^{0.05t} = 50$

18. $3e^{2x} - 5e^{x} + 2 = 0$

19. $9^{x} - 3^{x} - 2 = 0$

20. $1500\left(1 + \frac{0.05}{4}\right)^{4t} = 3000$

21. Write a word problem that would match the equation. What exponential growth problem can the equation above represent?

    $200(2^{4t}) = 5000$

22. Write a word problem that would match the equation. What exponential growth problem can the equation above represent?

23. A car is purchased for $28,500. After each year the resale value decreased by 35%. What will be the resale value be after 4 years? Round your answer to the nearest dollar. (Write your final answer in a sentence.)

24. A loan of $39,000 is made at 5% interest, compounded annually. After how many years will the amount due reach $63,000 or more? (Use a calculator if necessary.) Write the smallest possible whole number answer.

25. The number of bacteria in a certain population increases according to a continuous exponential growth model, with a growth rate parameter of 4.1% per hour. How many hours will it take for the sample to double?

Note: This is a continuous growth model.

Do not round any intermediate computations, and round your answer to the nearest whole hundredth.

26. An initial amount of $1800 is invested in an account at an interest rate of 2% per year compounded continuously. Find the amount in the account after 6 years. Round your answer to nearest cent.

27. Extra Credit: You want to buy a house that might be in range of $180,000 to $200,000 in 10 years. You know you will need to save at least a 20% down payment cost. How much should you invest per month starting now in an account that pays 5% interest so that in 10 years you will have enough for your down payment at that time? Please explain carefully how you computed this amount.