2.5 Finding x-intercepts of Polynomial and Rational Functions

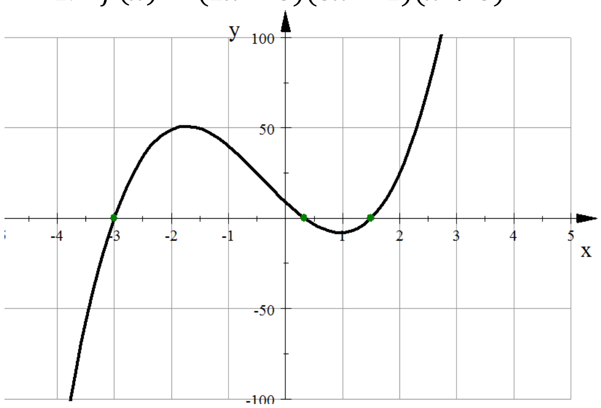
We have seen that the key to successfully graphing a rough sketch of the polynomial and rational functions was to be able to find the x-intercepts. For rational functions we also had to find all the asymptotes. We definitely used the zero product property to get the x-intercept.

Zero Product Property: The product of two real numbers A, and B equals zero if and only if one of the numbers must be zero. In other words, $AB = 0$, if and only if either $A = 0$, or $B = 0$.

Playing

Let's look at some the familiar graphs of odd and even degree polynomial functions to see what conclusions we can draw.

1. $f(x) = (2x - 3)(3x - 1)(x + 3)$

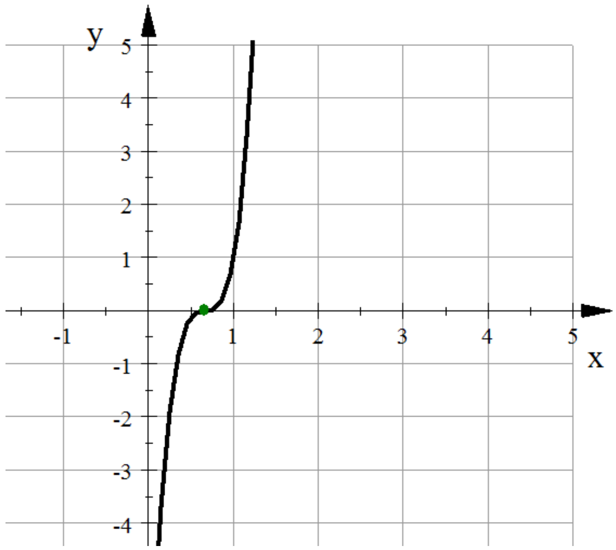


You can see this third degree polynomial has three real zeros. Also if you write the function in expanded form by multiplying all terms out we get $f(x) = (2x - 3)(3x - 1)(x + 3) = 6x^3 + 7x^2 - 30x + 9$
Remember that the x-intercepts are found by setting each factor to zero. So the x-intercepts here are $x = \frac{3}{2}, -\frac{1}{3}, \ -3 = -\frac{3}{1}$

We can also see that the function in the expanded form has a leading coefficient of 6 which is actually the product of all the leading coefficients of each of the individual terms in the product. Similarly the constant term of 9 is actually product of all the constant terms of each of the individual terms in the product. So looking at our rational zeros we notice that the numerator is a factor of the constant term in the expanded polynomial and the denominator is a factor of the constant term in the expanded polynomial.

Also note that the degree of the polynomial is three and we had three zeros. Clearly that is the maximum number of zeros or x-intercepts we can expect.

2. $f(x) = (3x - 2)^3$



You can see this third degree polynomial has three real zeros. Also if you write the function in expanded form by multiplying all terms out we get $f(x) = (3x - 2)^3 = 27x^3 - 54x^2 + 36x - 8$
Remember that the x-intercepts are found by setting each factor to zero. So the x-intercepts here are $x = \frac{2}{3}$
We can also see that the function in the expanded form has a leading coefficient of 27 which is actually the product of all the leading coefficients of each of the individual terms in the product. Similarly the constant term of -8 is actually product of all the constant terms of each of the individual terms in the product. So looking at our rational zero we notice that the numerator is a factor of the constant term in the expanded polynomial and the denominator is a factor of the leading coefficient in the expanded polynomial.

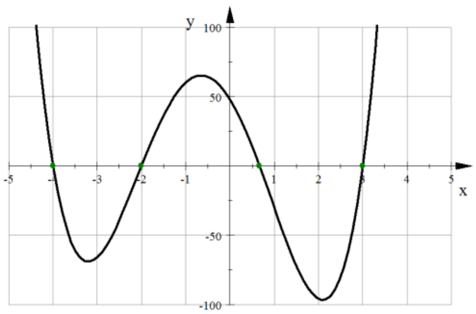
Also note that the degree of the polynomial is three and we have one real zero. Clearly that is the minimum number of zeros or x-intercepts we can expect given that odd functions have one end up and one end down for end behavior.

Observation: An polynomial function odd degree must have at least one real zero.

3. Let us look at some even degree polynomials to see what we can conclude with the zeros here.
$f(x) = (3x - 2)(x + 2)(x + 4)(x - 3) = 3x^4 + 7x^3 - 36x^2 - 52x + 48$

You can see this fourth degree polynomial has four real zeros the maximum possible. Remember that the x-intercepts are found by setting each factor to zero. So the x-intercepts here are $x = \frac{2}{3}, -2, -4, 3$

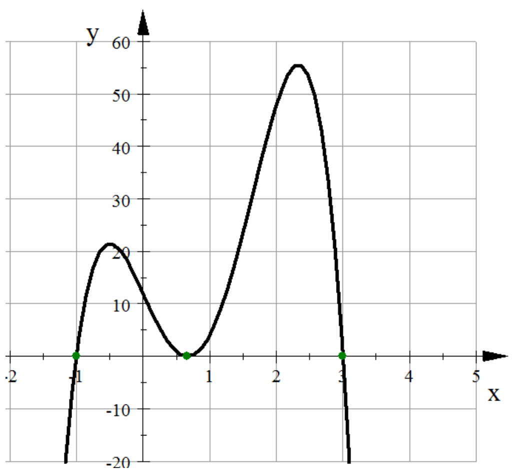
We can also see that the function in the expanded form has a leading coefficient of 3 which is actually the product of all the leading coefficients of each of the individual terms in the product. Similarly the constant term of 48 is actually product of all the constant terms of each of the individual terms in the product. So looking at our rational zeros we notice that the numerator is a factor of the constant term in the expanded polynomial and the denominator is a factor of the constant term in the expanded polynomial.



4. Let us look at some even degree polynomials to see what we can conclude with the zeros here.
$f(x) = (3x - 2)^2(-x - 1)(x - 3) = -9x^4 + 30x^3 - x^2 - 28x + 12$

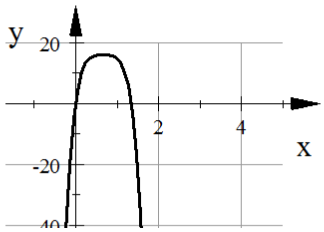
You can see this fourth degree polynomial has three distinct real zeros. So the x-intercepts here are $x = -1,\ 3$, and $\frac{2}{3}$ of multiplicity 2 (that means that zero repeats itself twice)

We can also see that the function in the expanded form has a leading coefficient of -9 which is actually the product of all the leading coefficients of each of the individual terms in the product. Similarly the constant term of 12 is actually product of all the constant terms of each of the individual terms in the product. So looking at our rational zeros we notice that the numerator is a factor of the constant term in the expanded polynomial and the denominator is a factor of the constant term in the expanded polynomial.



5. Many other examples are below

$f(x) = -(3x-2)^4 + 16$
$= -81x^4 + 216x^3 - 216x^2 + 96x$



This function has two real zeros and similar observations as before can be made about its zeros. However if you solve $-(3x-2)^4 + 16 = 0$ we get

$-(3x-2)^4 = -16$

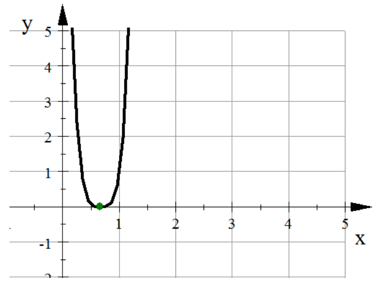
$(3x-2)^4 = 16$

$3x - 2 = 2$ or $3x - 2 = -2$

$x = \frac{4}{3}$, or $x = 0$ are the two real zeros.

What about the other two zeros?

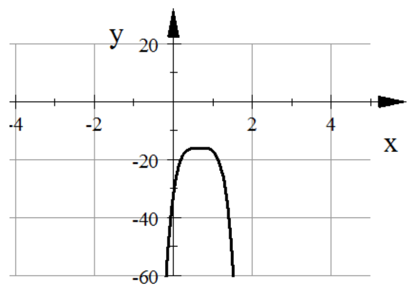
$f(x) = (3x-2)^4$
$= -81x^4 - 216x^3 + 216x^2 - 96x + 16$



This function has one real zero of multiplicity four and similar observations as before can be made about its zeros.

$f(x) = -(3x-2)^4 - 16$
$= -81x^4 + 216x^3 - 216x^2 + 96 - 32$

This function has no real zeros



We can summarize our observations above as follows:

Rational Zeros Theorem:

If the polynomial function with integer coefficients has rational zeros say $x = \frac{p}{q}$ then $p =$a factor of the constant term in the polynomial, and $q =$ a factor of the leading coefficient.

This will give us a rough start to check potential rational zeros. We also know we can do long division to find the actual zeros. We will introduce synthetic division to be a little more efficient.

So let's if we can find the real zeros of the polynomial functions below. A more in-depth look at finding all zeros of certain functions will be looked at in the next chapter.

Long division of polynomials is similar to the whole number long division as we saw when finding oblique asymptotes. See example below for $6x^2 + 7x + 4 \div \mathrm{x} + 2$. As you can see in long division since the first term subtracts out the second term is the one that is useful and so when dividing a polynomial with terms like $x - a$ we can use something called synthetic division. You use a from the divisor $x - a$ and then instead of subtracting we add as shown below.

Step 1: The coefficients of the dividend polynomial go in the boxes in the top row which for us would be the 6,7,6.

Step 2: The number $a = -2$ goes in the second row to the right most.

Step 3: Then bring the coefficient of x^2 term down as is in the last row.

Step 4: Multiply the first number in the bottom row and put the result in the second row under 7, add the two terms in that column and put that resulting number in the last row in that column, multiply that number by -2 and put that number in the second row last column and continue the process. The last number in the last row is the remainder. The remaining numbers are the coefficient of the quotient which is a polynomial of one degree lower than the dividend.

As you can see below, synthetic division helps and can be used when finding zeros.

Long Division

$$\begin{array}{r|l} & 6x - 5 \\ \hline (x+2) & 6x^2 + 7x + 4 \\ & -(6x^2 + 12x) \\ \hline & \quad -5x + 4 \\ & \quad -(5x - 10) \\ \hline & \qquad 14 \end{array}$$

Quotient $6x - 5$

Remainder is 14

Synthetic Division

	6	7	4
-2		-12	10
	6	-5	14

Quotient $6x - 5$

Remainder is 14

Cover Sheet for Sec 2.5 Name: ______________________________

Please write your observations by filling the blanks below.

1. What is the maximum number of x-intercepts a polynomial of degree n can have?

2. Why does an odd degree polynomial have at least one real root or one x-intercept?

3. If a zero or a polynomial of degree n has multiplicity of n, what does that mean?

4. If you have a polynomial of degree 4 with no real zeros then, what can you say about it?

5. How could the “Rational Zeros Theorem” help in finding zeros of a polynomial?

Video Log 2.5

Find all the real zeros or the x-intercepts of the function below and then give a rough sketch.

1. $f(x) = 3x^3 + 2x^2 - 3x - 2$

2. $f(x) = 30x^4 + 39x^3 - 54x^2 - 21x + 6$

3. $f(x) = 2x^4 + x^3 - 7x^2 - 4x - 4$
   After you give a rough sketch, plot your graph using a graphing utility to see if it is accurate. How do you explain the difference.