2.4 Graphing Rational Functions

We introduced the rational functions briefly in chapter 1. Below is a recall of what was discussed.

Rational Function: A rational function is defined as the ratio of two polynomial functions denoted as $R(x) = \frac{P(x)}{Q(x)}$, for polynomial function $P(x)$, $Q(x)$ and valid for real numbers x for which $Q(x) \neq 0$.

Domain of $R(x)$ = all real numbers where $Q(x) \neq 0$

Asymptote: A line $y = ax + b$ is an asymptote to a function $y = f(x)$ if and only if the distance between it and the function goes to zero as they tend to infinity.

There can be several types of asymptotes.

Vertical Asymptote: A vertical line of the form $x = a$ is called a vertical asymptote to a function $y = f(x)$ if and only if the function $f(x)$ approaches either ∞ or $-\infty$ as x approaches a either from the left or right.

Horizontal or Oblique Asymptote: A line $y = mx + b$ is called a horizontal (when $m = 0$) or oblique ($m \neq 0$) asymptote to a function $y = f(x)$ if and only if the distance between the function $f(x)$ and $y = mx + b$ approaches zero as x approaches either ∞ or $-\infty$.

Let us review a part of the example of plotting a rational function $R(x) = \frac{1}{x}$ we looked at in chapter 1.

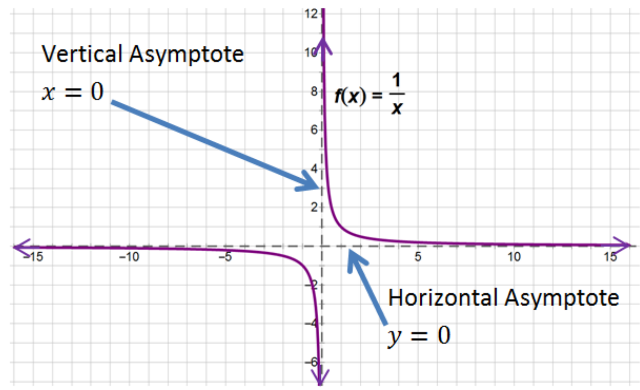
Example

1. $R(x) = \frac{1}{x}$
   a. Sketch the graph of $R(x)$.
   b. What is domain of this function?

Play with it to see if you can recall what we did before looking on the next page.

To sketch the graph of a function we do not know much about for now involves plotting points. As we can see the denominator cannot be zero.

x	$y = \frac{1}{x}$		x	$y = \frac{1}{x}$
-0.1	$\frac{1}{-0.1} = -10$		-10	$\frac{1}{-10} = -0.1$
-0.01	$\frac{1}{-0.01} = -100$		-100	$\frac{1}{-100} = -0.01$
-0.001	$\frac{1}{-0.001} = -1000$		-1000	$\frac{1}{-1000} = -0..001$
0.1	$\frac{1}{0.1} = 10$		10	$\frac{1}{10} = 0.1$
0.01	$\frac{1}{0.01} = 100$		100	$\frac{1}{100} = 0.01$
0.001	$\frac{1}{0.001} = 1000$		1000	$\frac{1}{1000} = 0..001$



Domain: All nonzero real numbers

From the chart we can see that the closer to zero the x values come from left or the right of zero, the y values either shoot to ∞, or $-\infty$ respectively. In other words, if we were to zoom in closer and closer to $x = 0$, the graph resembles the vertical line $x = 0$. Making the line $y = 0$ the horizontal asymptote and the line $x = 0$ the vertical asymptote for the rational function $f(x) = \frac{1}{x}$.

Playing

We can generalize our observations from plotting the function $R(x) = \frac{1}{x}$. Note that for the function $f(x) = \frac{1}{x^n} \to 0$ when $x \to \pm\infty, n$ a positive integer. Another notation mathematicians use to say the statement above is using a limit notation which if you continued forward to calculus you will see and it will look like

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \qquad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

Also note that

a) When n is even, $f(x) = \frac{1}{x^n} \to \infty$ when $x \to 0$ from either side.

b) When n is odd, $f(x) = \frac{1}{x^n} \to \infty$ when $x \to 0$ from the right and $f(x) = \frac{1}{x^n} \to -\infty$ when $x \to 0$ from the left .

So for the function $f(x) = \frac{1}{x^n}$ the vertical asymptote is at $x = 0$, and the horizontal asymptote is at $y = 0$.

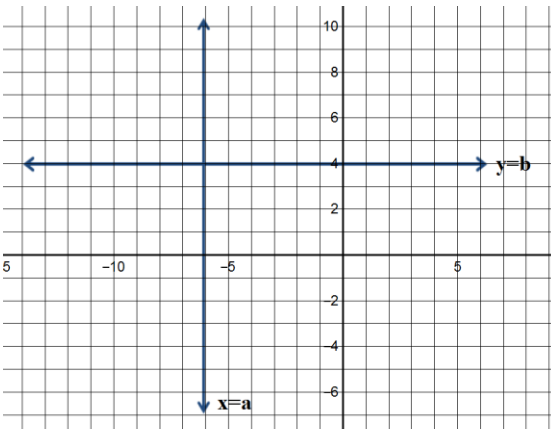
Using our knowledge from transformation of functions we can see that for the function

1. $g(x) = \frac{1}{(x-a)^n}$, the vertical asymptote is at $x = a$, and the horizontal asymptote is at $y = 0$.
2. $g(x) = \frac{1}{x^n} + b$, the vertical asymptote is at $x = 0$, and the horizontal asymptote is at $y = b$.
3. $g(x) = \frac{1}{(x-a)^n} + b$, the vertical asymptote is at $x = a$, and the horizontal asymptote is at $y = b$.
4. $g(x) = \frac{1}{(x-a)^n} + ax + b$, the vertical asymptote is at $x = a$, and the oblique asymptote is at $y = ax + b$. The reason oblique asymptote is $y = ax + b$ is the fact that $\frac{1}{(x-a)^n} \to 0$ when $x \to \pm\infty$ making the distance between $g(x) = \frac{1}{(x-a)^n} + ax + b$ and $y = ax + b$ go to zero when $x \to \pm\infty$.

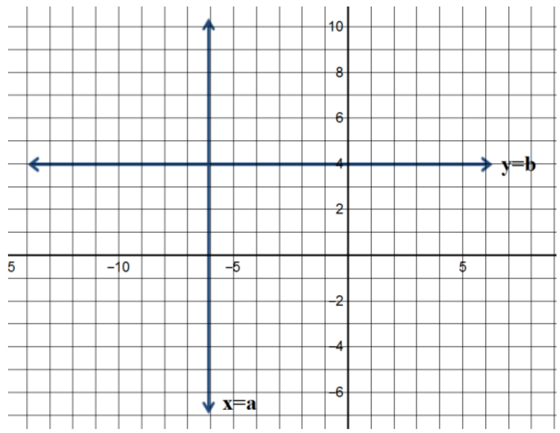
Let's take a look at all the scenarios of what a graph may look like when you have the asymptotes of a rational function. We want you to explore all possibilities of what the graph of a rational function may do if you knew the asymptotes.

Sketch the possibilities below for how rational function might behave toward the different asymptotes you see below.

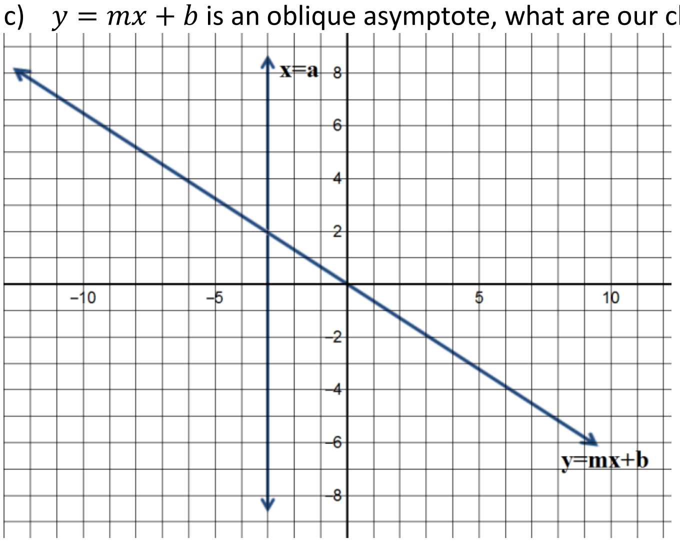
a) $y = b$ is a horizontal asymptote, what are our choices?



b) x= a is a vertical asymptote, what are our choices?



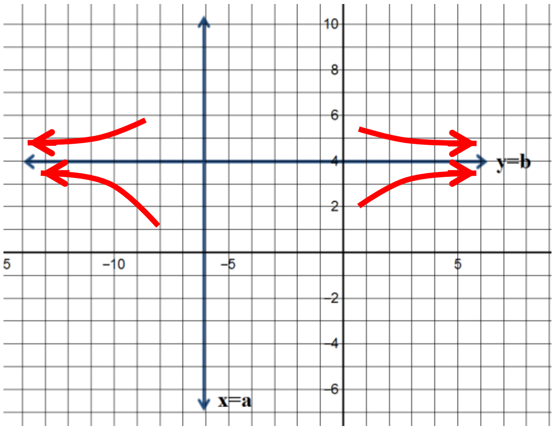
c) $y = mx + b$ is an oblique asymptote, what are our choices?

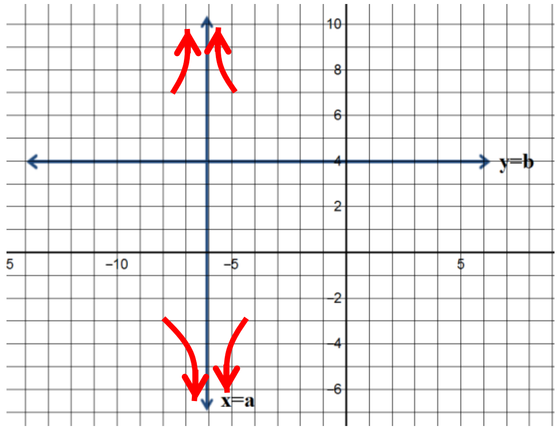


Do you think you exhausted all possibilities? How sure are you?

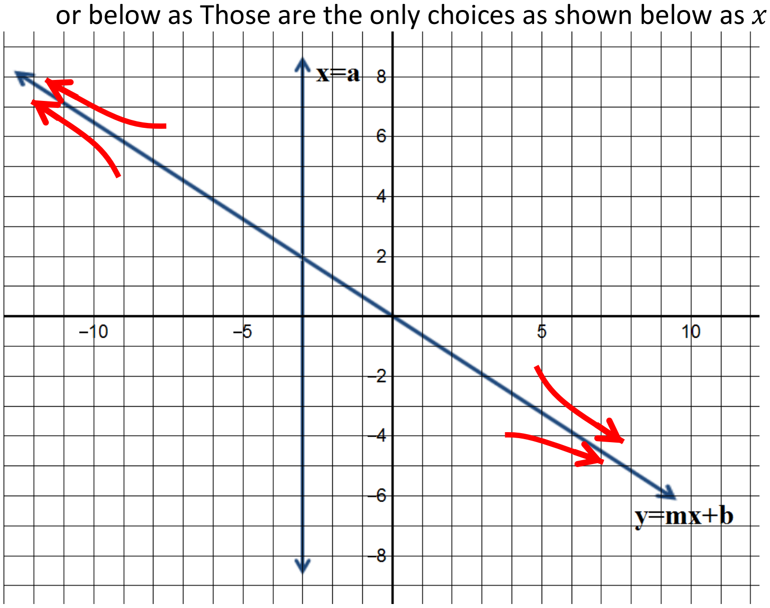
a) For the horizontal asymptote $y = b$ the graph of the function eventually must approach the line either from the above or the below. Those are the only choices as shown below.

b) For the vertical asymptote $x = a$ the graph of the function must approach this line from the left or right and shoots up or down. Those are the only choices as shown below.





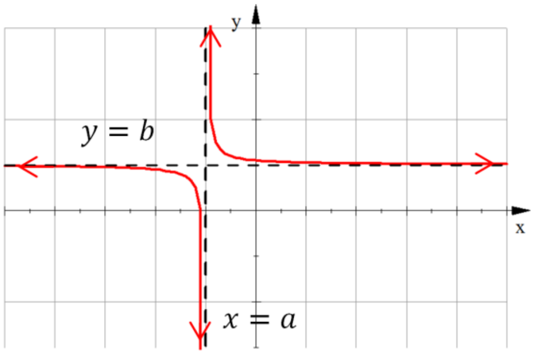
c) For the oblique asymptote $y = mx + b$ the graph must get closer and closer from above or below as Those are the only choices as shown below as $x \to \pm\infty$.



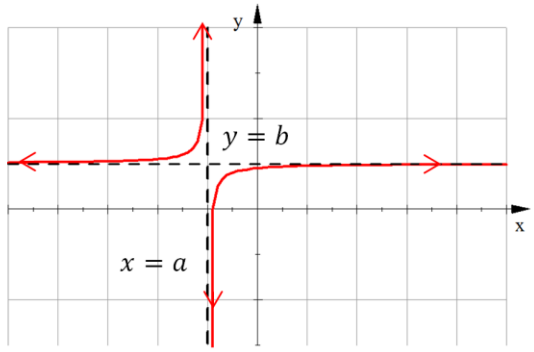
So now lets us look at how we can put these observations together to imagine the different possible graphs we can have with given asymptotes.

Below are some possible graphs where $x = a$ is the vertical asymptote, and $y = b$ is the horizontal asymptote.

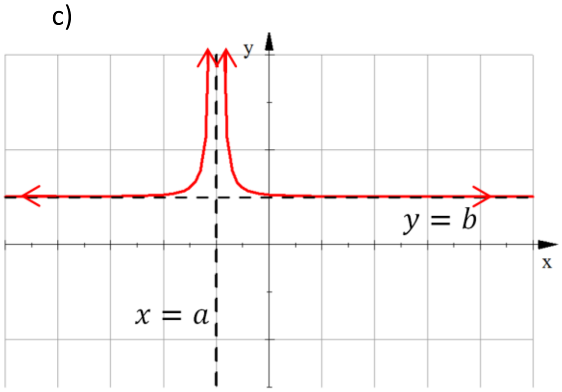
a)



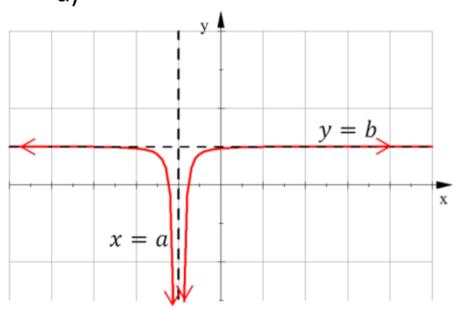
b)



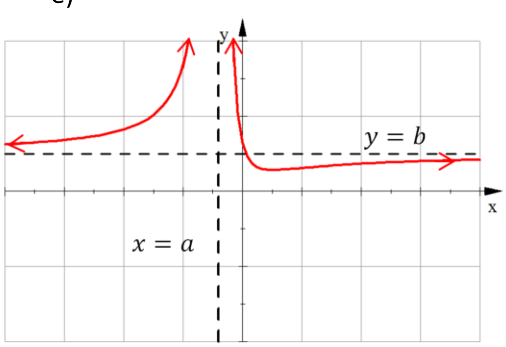
c)



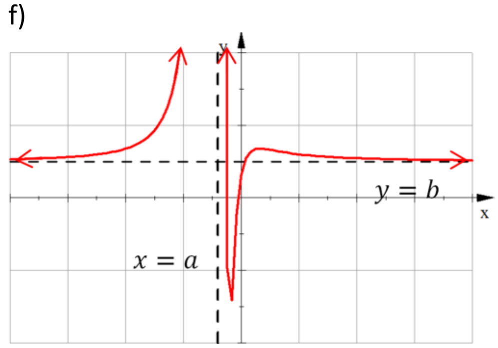
d)



e)



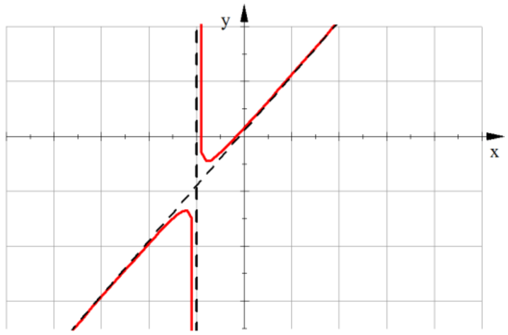
f)



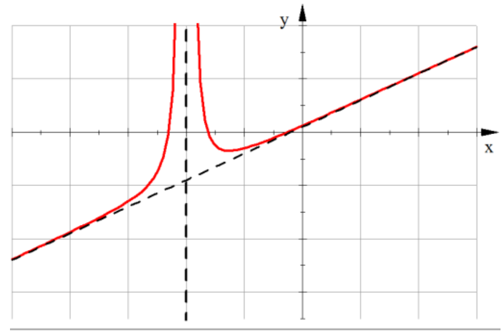
As you can see there are countless possibilities. The last two may seem not possible but there can be finite intersections of the graph of the rational function and the horizontal asymptotes. As long as eventually the graph gets closer and closer to the line $y = b$. You can also see that the left side of the graph in the last two examples gets to the vertical asymptote a little slower than the right side. So it is actually a lot of fun to graph these. Remember this means like in graphing polynomial functions if we find the asymptotes and look, intercepts, where the graph hits the horizontal asymptotes, and then looking at the sign chart in the right intervals should get us a good rough sketch.

Below are some possible graphs where $x = a$ is the vertical asymptote, and $y = mx + b$ is the horizontal asymptote.

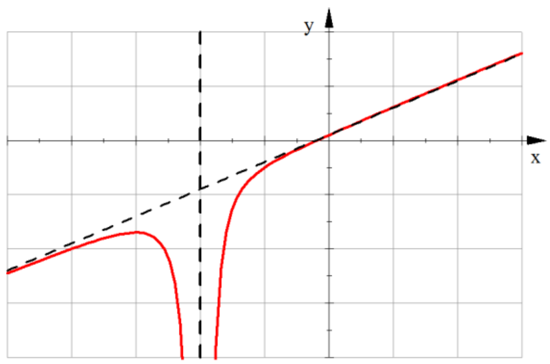
a)



b)



c)



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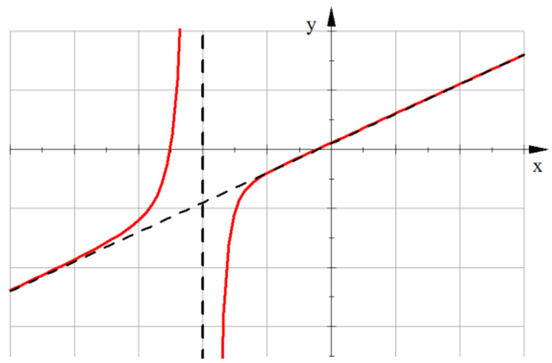

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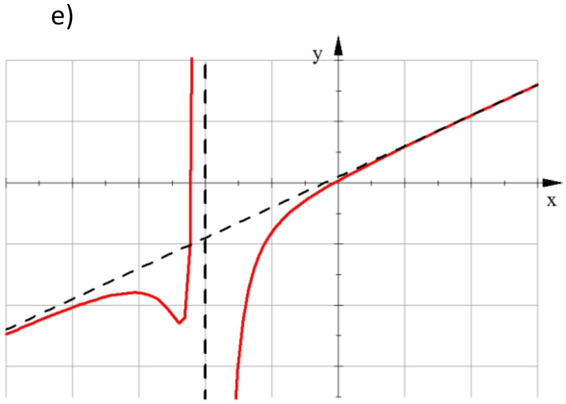

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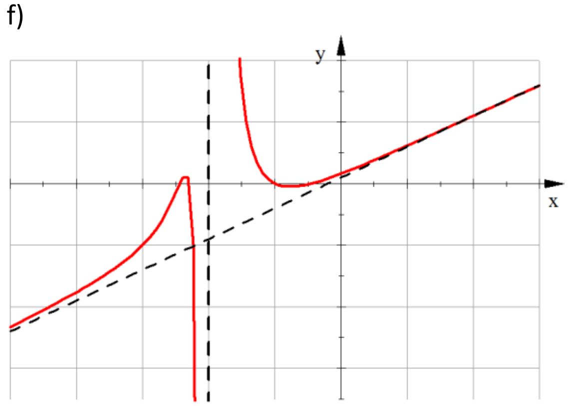


As you can see there are countless possibilities. Again the last two may seem not possible but there can be finite intersections of the graph of the rational function and the oblique asymptotes. As long as eventually the graph gets closer and closer to the line $y = mx + b$. So again remember this means like in graphing polynomial functions if we find the asymptotes and look, intercepts, where the graph hits the oblique asymptotes, and then looking at the sign chart in the right intervals should get us a good rough sketch.

Steps to graph a rational function

Let $R(x) = \frac{P(x)}{Q(x)}$ be a rational function in lowest terms then to graph it we can follow some standard steps to get our asymptotes and then plot a rough sketch.

- Step 1 Do long division and write $R(x) = \frac{P(x)}{Q(x)} = \frac{remainder}{Q(x)} + Quotient.$
- Step 2 Set denominator $Q(x) = 0$ to find all the vertical asymptotes
- Step 3 The equation $y = Quotient$ will give you the horizontal or oblique asymptotes.
- Step 4 Setting the numerator $P(x) = 0$ will give you all of your x-intercepts.
- Step 5 Setting $x = 0$ in the function $R(x)$ will give you your y-intercepts.
- Step 6 Set $R(x) = Quotient$ to see where the function intersects the horizontal or oblique asymptotes if any. That means setting the $remainder = 0$ to get the points of intersection.
- Step 7 Use the x-intercepts and the x values of the vertical asymptote as your break points and do the sign chart to determine where the function is positive and negative. Then use the information to sketch the rough graph of the rational function.

Use these steps to solve the following problems.

Practice Problems

The first one is done for you, please attempt the others before class.

For all the rational functions below please find the requested information below and then graph the function. In your graph show all the relevant information and label all parts of the graph. Draw the asymptotes if any as dotted lines.

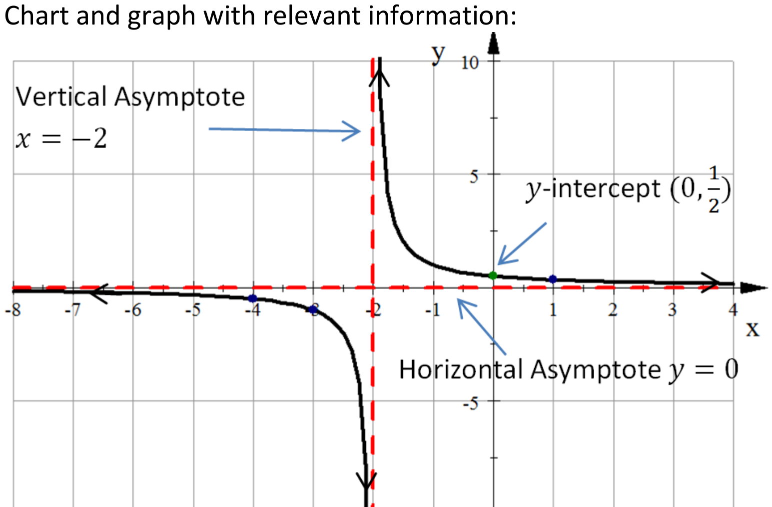
a) Domain of the function
b) Perform long division and write the rational function $\frac{Numerator}{Denominator} = \frac{remainder}{Denominator} + Quotient.$
c) All the vertical and horizontal/oblique asymptotes.
d) All the x-intercepts and y-intercept if possible.
e) All the points if any where the graph intersects the horizontal or oblique asymptotes.
f) Show the chart that determines all the x-values where the function takes on negative or positive values. Plot two points on either side of the vertical asymptotes and any additional points as needed to finally sketch your graph

1. $f(x) = \frac{1}{x+2}$

a) Domain of the function: All real numbers except $x = -2$.	Function rewrite as $\frac{remainder}{Denominator} + Quotient$ $f(x) = \frac{1}{x+2} + 0$
b) Horizontal or Oblique Asymptote Horizontal Asymptote is $y = 0$	Vertical Asymptotes: $x = -2$
c) x-intercepts: None since $\frac{1}{x+2} \neq 0$ for any x.	y-intercept $y = f(0) = \frac{1}{0+2} = \frac{1}{2}$

d) Points of intersection with the horizontal or oblique asymptotes:
$\frac{1}{x+2} + 0 = 0$ has no solutions, so no points of intersection.

e) Sign Chart and graph with relevant information:



x	y
-4	$\frac{1}{-4+2} = -\frac{1}{2}$
-3	$\frac{1}{-3+2} = -1$
0	$\frac{1}{0+2} = \frac{1}{2}$
1	$\frac{1}{1+2} = \frac{1}{3}$

Sign Chart	$x < -2$	$x > 2$
$\frac{1}{x+2}$	$\frac{1}{-3+2} = -1 < 0$	$\frac{1}{1+2} = \frac{1}{3} > 0$

2. $S(x) = \frac{3x^2+4x+1}{x+2}$

Before we can answer anything let us do long division so it will allows to find the oblique and horizontal asymptotes easily

$$\begin{array}{r|lll} & 3x & -2 & \\ x+2 & 3x^2 & +4x & +1 \\ - & (3x^2 & +6x) & \\ \hline & & -2x & +1 \\ & - & (-2x & -4) \\ \hline & & & 5 \end{array}$$

Long division gives us

$$S(x) = \frac{3x^2+4x+1}{x+2} = \frac{5}{x+2} + 3x - 2$$

a) Domain of the function: All real numbers except for $x = -2$.	Function rewrite as $\frac{remainder}{Denominator} + Quotient$ $S(x) = \frac{3x^2+4x+1}{x+2} = \frac{5}{x+2} + 3x - 2$
b) Horizontal or Oblique Asymptote Oblique Asymptote is $y = 3x - 2$	Vertical Asymptotes: $x = -2$
c) x-intercepts: $\left(-\frac{1}{3}, 0\right)$, or $(-1,0)$ $\frac{3x^2+4x+1}{x+2} = 0$ or $3x^2 + 4x + 1 = (3x+1)(x+1) = 0$ giving us $x = -\frac{1}{3}$ or $x = -1$	y-intercept $y = S(0) = \frac{3(0)^2+4(0)+1}{0+2} = \frac{1}{2}$

d) Points of intersection with the horizontal or oblique asymptotes: $\frac{5}{x+2} + 3x - 2 = 3x - 2$ or $\frac{5}{x+2} = 0$ has no solutions, so no points of intersection.

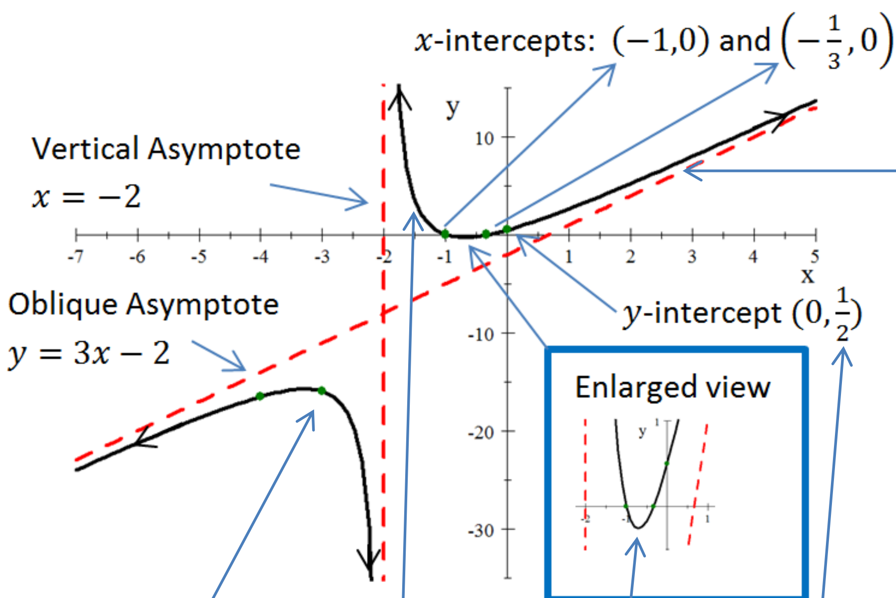
e) Sign Chart and graph with relevant information:

Plot two points to graph the horizontal asymptote $y = 3x - 2$.

x	$y = 3x - 2$
0	-2
2	4

Plot two points on either side of the vertical asymptote.

x	y
-4	$\dfrac{3(-4)^2 + 4(-4) + 1}{-4 + 2} = -\dfrac{33}{2}$
-3	$\dfrac{3(-3)^2 + 4(-3) + 1}{-3 + 2} = -16$
0	$\dfrac{3(0)^2 + 4(0) + 1}{0 + 2} = \dfrac{1}{2}$
1	$\dfrac{3(1)^2 + 4(1) + 1}{1 + 2} = \dfrac{8}{3}$



To really see the values you must check the sign chart for when $x < -2$, $-2 < x < -1$, $-1 < x < -\frac{1}{3}$ and $-\frac{1}{3} < x < 2$ and $x > 2$

Sign chart	$x < -2$	$-2 < x < -1$	$-1 < x < -\frac{1}{3}$	$-\frac{1}{3} < x < 2$	$x > 2$
Test Point	-3				
$S(x)$	$S(-3) = -16 < 0$	$S(-1.5) = 3.5 > 0$	$S(-0.5) = -\frac{1}{6} < 0$	$S(0) = \frac{1}{2} > 0$	$S(3) = 8 > 0$

3. $g(x) = \frac{5x^2+4x-7}{x^2-4}$

Before we can answer anything let us do long division so it will allows to find the oblique and horizontal asymptotes easily

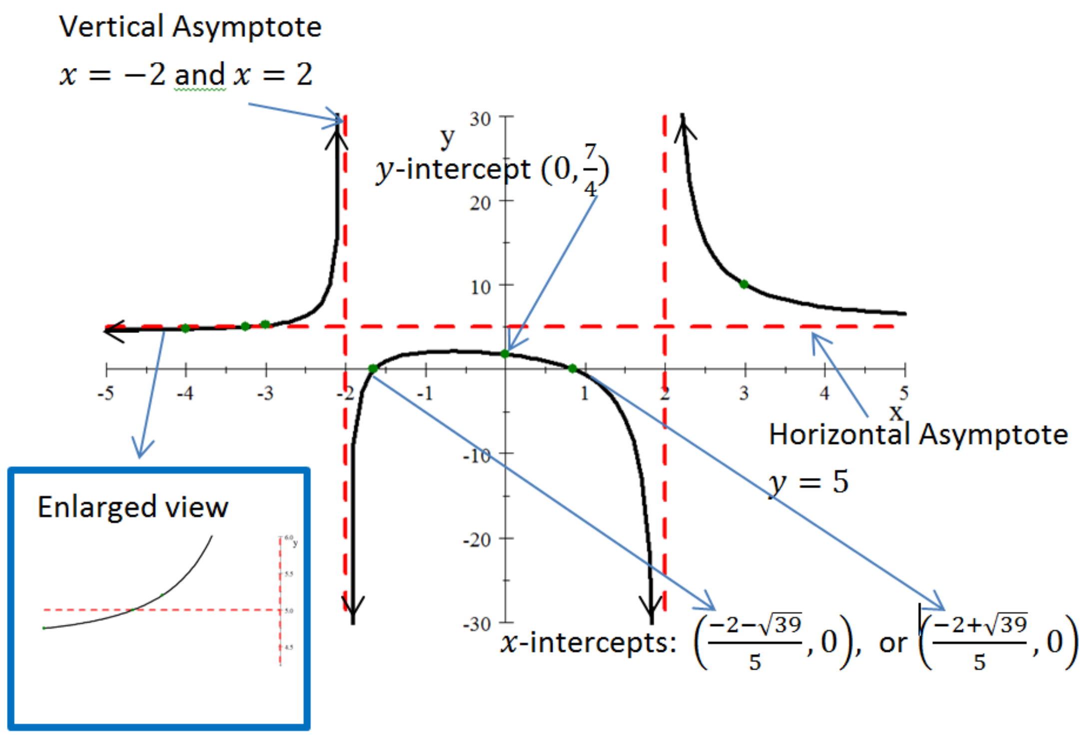
$$\begin{array}{r|rrr} & & & 5 \\ \hline x^2-4 & 5x^2 & +4x & -7 \\ - & (5x^2 & & -20) \\ \hline & & 4x & +13 \end{array}$$

Long division gives us

$$g(x) = \frac{5x^2+4x-7}{x^2-4} = \frac{4x+13}{x^2-4} + 5$$

a) Domain of the function: All real numbers except for $x = -2,2$. Since $x^2 - 4 = 0$ at $x = -2,2$.	Function rewrite as $\frac{remainder}{Denominator} + Quotient$ $g(x) = \frac{5x^2+4x-7}{x^2-4} = \frac{4x+13}{x^2-4} + 5$
b) Horizontal or Oblique Asymptote Horizontal Asymptote is $y = 5$	Vertical Asymptotes: $x = -2$, and $x = 2$
c) x-intercepts: $\left(\frac{-2-\sqrt{39}}{5}, 0\right)$, or $\left(\frac{-2+\sqrt{39}}{5}, 0\right)$ $\frac{5x^2+4x-7}{x^2-4} = 0$ or $5x^2 + 4x - 7 = 0$ use quadratic formula or completing the squares to solve this equation. $x = \frac{-4 \pm \sqrt{(4)^2 - 4(5)(-7)}}{2(5)}$ $= \frac{-4 \pm \sqrt{156}}{10} = \frac{-4 \pm 2\sqrt{39}}{10} = \frac{-2 \pm \sqrt{39}}{5}$ $x = \frac{-2-\sqrt{39}}{5} \approx -1.65$ or $x = \frac{-2+\sqrt{39}}{5} \approx 0.85$	y-intercept $y = g(0) = \frac{5(0)^2+4(0)-7}{(0)^2-4} = \frac{7}{4}$ d) Points of intersection with the horizontal or oblique asymptotes: $\frac{4x+13}{x^2-4} + 5 = 5$ Or $\frac{4x+13}{x^2-4} = 0$ or $4x + 13 = 0$ When $x = -\frac{13}{4} \approx -3.25$

e) Sign Chart and graph with relevant information:
Make your own sign chart and plot relevant points to verify the graph below.



Cover Sheet for Sec 2.4 Name: ________________________________

Please write your observations by filling the blanks below.

1. How do you determine the vertical asymptotes? Why does your method work?

2. Do you think the graph of rational function can intersect a vertical asymptote? Why or why not?

3. How do you determine the horizontal or oblique asymptotes?

4. What would a graph of the function $f(x) = \frac{1}{x} + x^2$ look like?

5. If the numerator and denominator have common factors in the rational function, how does that affect the graph of the rational function?

6. How do you find x-intercepts and y-intercepts of the rational function?

7. If you knew all the asymptotes of the rational function, would there be a unique function that corresponds to these asymptotes? What other information would you need?

8. List any difficulties you may have had in this section.

Video Log 2.4

For all the rational functions below please find the requested information below and then graph the function. In your graph show all the relevant information and label all parts of the graph. Draw the asymptotes if any as dotted lines.

a) Domain of the function

b) Perform long division and write the rational function $\frac{Numerator}{Denominator} = \frac{remainder}{Denominator} + Quotient.$

c) All the vertical and horizontal/oblique asymptotes.

d) All the x-intercepts and y-intercept if possible.

e) All the points if any where the graph intersects the horizontal or oblique asymptotes.

f) Show the chart that determines all the x-values where the function takes on negative or positive values. Plot two points on either side of the vertical asymptotes and any additional points as needed to finally sketch your graph

1. $g(x) = \frac{2x+1}{x-3}$

2. $R(x) = \frac{x+1}{(x+2)^2}$

3. $S(x) = \frac{3x^2+4x+1}{x+2}$

4. $f(x) = \frac{2x^2+1}{x-1}$

5. $T(x) = \frac{3x^2+4x+1}{(x+2)^2}$

6. $f(x) = \frac{1}{x^2-4}$

7. $f(x) = \frac{x+1}{x^2-4}$

8. $P(x) = \frac{3x^3+x^2+3x+1}{x^2-4}$

9. $S(x) = \frac{3x^3 - 3x - 1}{(x+1)^2}$

10. $H(x) = \frac{x^2-1}{x+1}$

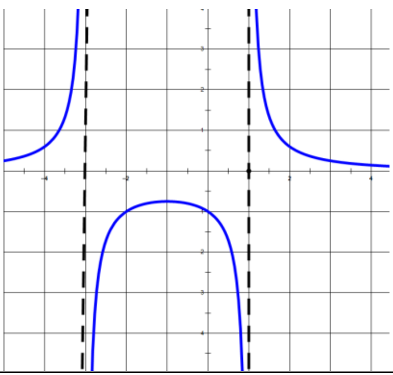
11. $R(x) = x^2 - \frac{1}{x}$

12. Choose the graphs below that match the graphs of the rational functions below.

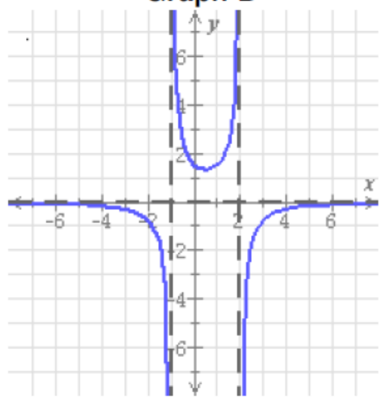
I. $g(x) = \frac{3x+3}{x^2+2x-3}$

II. $h(x) = \frac{3}{x^2+2x-3}$

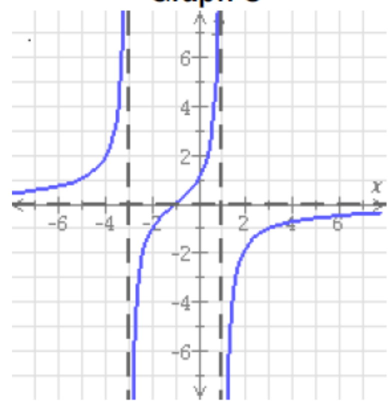
Graph A



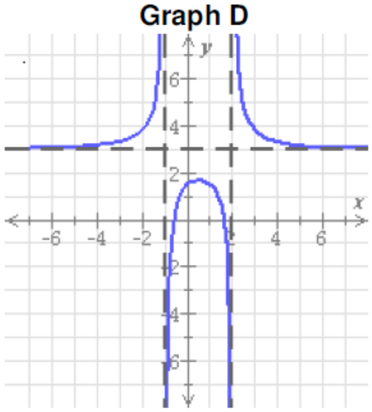
Graph B



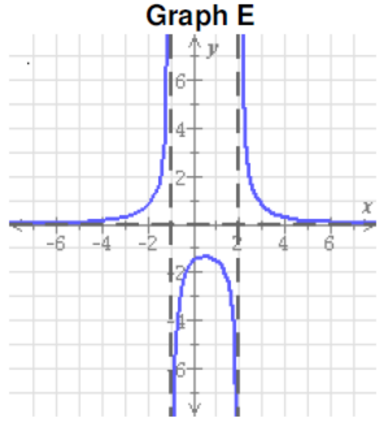
Graph C



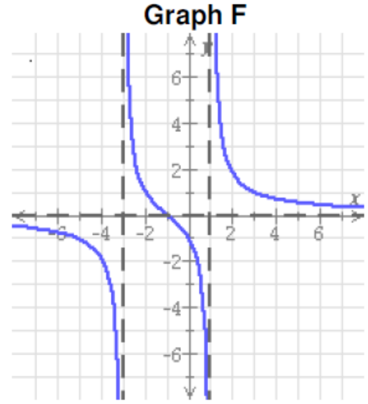
Graph D



Graph E



Graph F



13. The figure below shows the graph of a rational function f with vertical asymptotes at $x = 2, x = 6$, horizontal asymptote of $y = -2$. The graph also has x-intercepts of 3 and -4 and it passes through the point $(0,2)$. The equation of $f(x)$ has one of the five forms shown below. Choose the appropriate form for $f(x)$ and then write the equation. You can assume that $f(x)$ is in the simplest form.

- ☐ $f(x) = \frac{a}{x-b}$
- ☐ $f(x) = \frac{a(x-b)}{x-c}$
- ☐ $f(x) = \frac{a}{(x-b)(x-c)}$
- ☐ $f(x) = \frac{a(x-b)}{(x-c)(x-d)}$
- ☐ $f(x) = \frac{a(x-b)(x-c)}{(x-d)(x-e)}$