2.2 Graphing Conic Sections

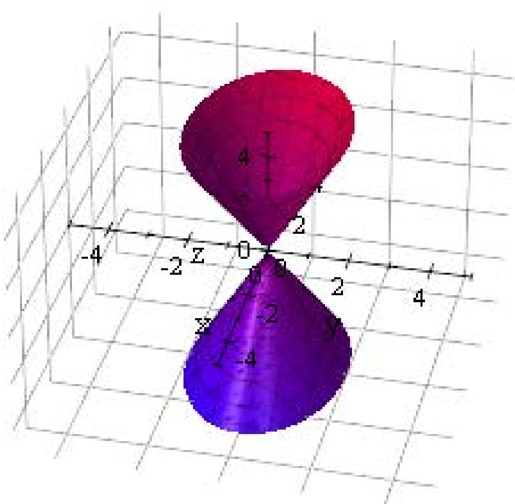
- Conic Sections Part 1 (17 min) https://www.youtube.com/watch?v=UZlB9Bs8hQg
- Conic Section Part 2 (12 min) https://www.youtube.com/watch?v=ofZn7v1jJ0U
- Part 3 Equations of Circles (10 mins) http://www.youtube.com/watch?v=fzNXmoCHRCk

Conic Sections: Conic sections are the curves formed by the intersection of a plane and a cone. A mathematical cone can be imagined as two ice-cream cones attached at the base as shown below.

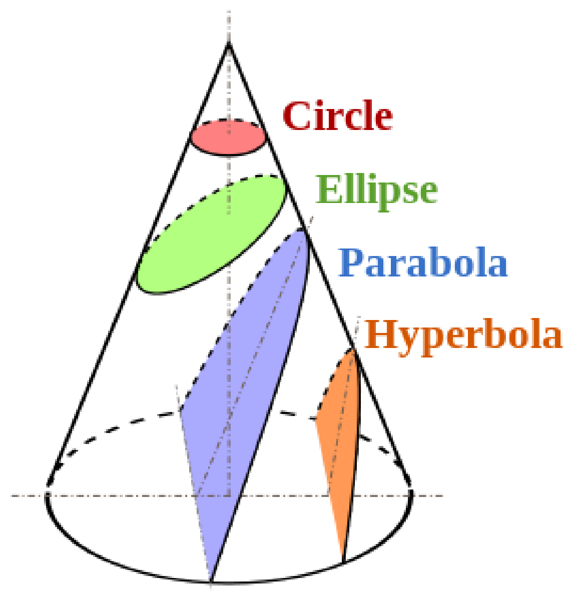
A conic is the "Locus" (which is another word for "path“) of a point P in the plane so that its distance from a fixed point F (called the "focus") in the plane has a constant ratio "e" (called the "eccentricity") to its distance from a fixed straight line d (called the "directrix") in the plane.

- $e = 1$ the conic is called a parabola
- $e < 1$ the conic is called an ellipse
- $e > 1$ the conic is called a hyperbola.

Mathematical Cone



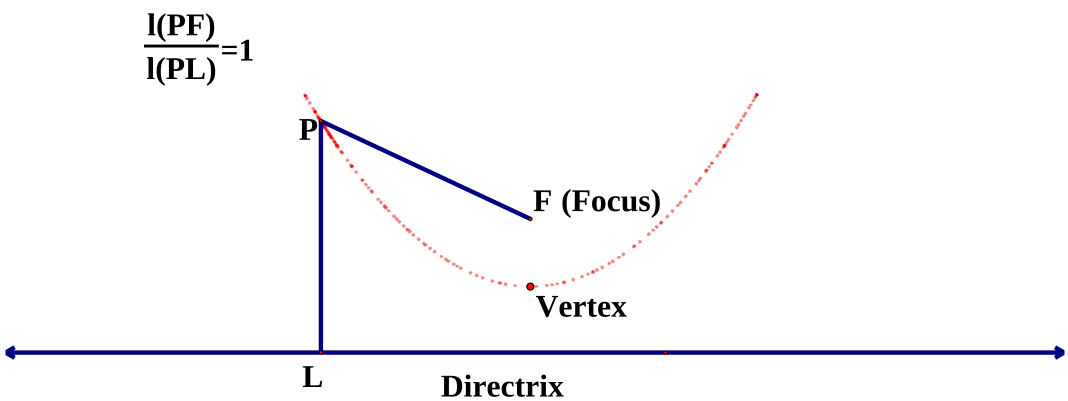
Intersections of a plane and a cone.



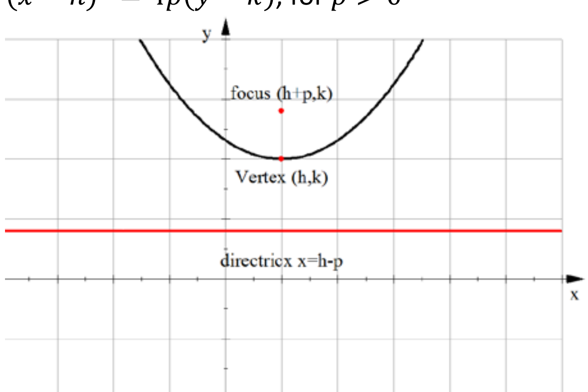
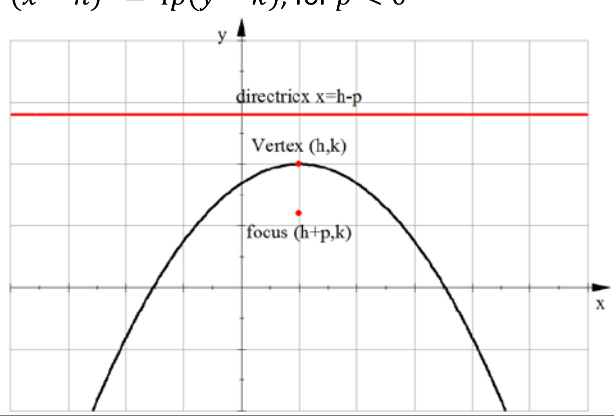
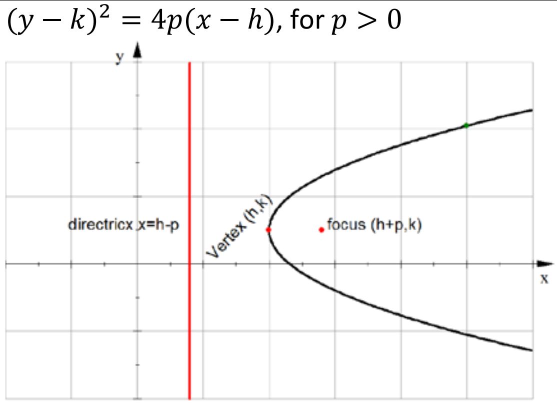
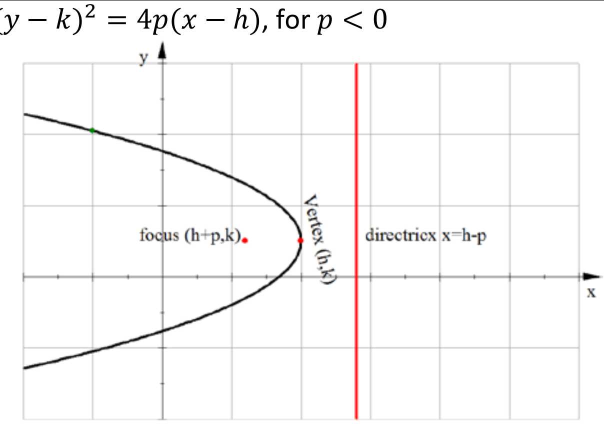
2.2a Parabolas

Parabola: A parabola is collection of all points equidistant from a fixed point (focus) and the fixed line (directrix).

Consider a fixed point and line as shown below. The locus or collection of all points equidistant from the fixed point focus and the fixed line directrix are shown below. You may notice this shape as you have seen it before in form of quadratic function in one variable.



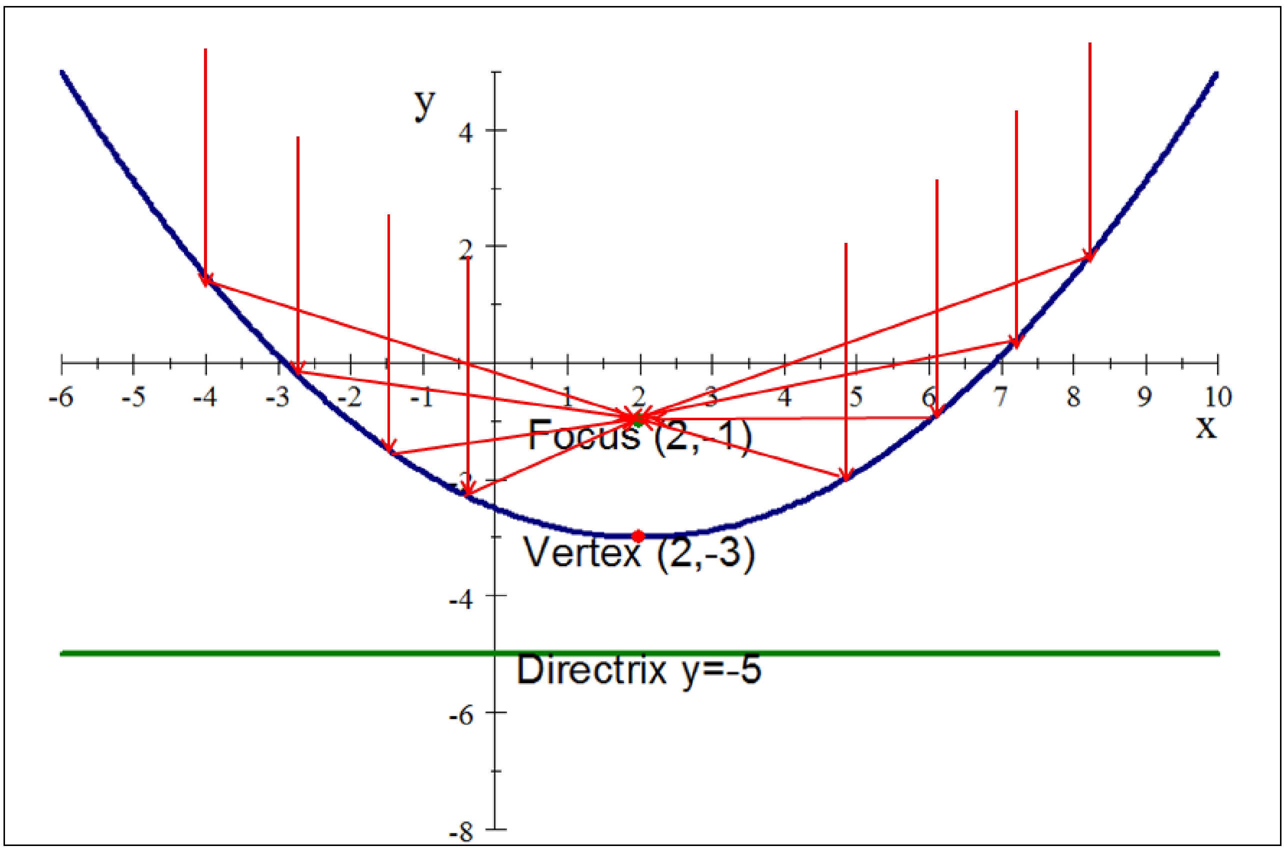
The distance PL and PF are the same in a parabola. You can see from the picture that the collection of all points (shown red here) form the parabola.
In general then,

$(x-h)^2 = 4p(y-k)$, for $p > 0$ 	$(x-h)^2 = 4p(y-k)$, for $p < 0$ 
$(y-k)^2 = 4p(x-h)$, for $p > 0$ 	$(y-k)^2 = 4p(x-h)$, for $p < 0$ 

Consider the parabola shown below.
$(x-2)^2 = 8(y+3)$, $4p = 8$ which gives us $p = 2$. Vertex is $(2,-3)$, directrix will be at $y = -5$, and focus is at $(2,1)$.

Reflective Property of Parabola: One of the properties of a parabola is that if an energy source located far away transmits energy and the parabola is made of a reflective surface the energy bounces of the parabola and merges onto the focus. Similarly if an energy source is located at the focus it will bounce off the parabola along the lines perpendicular to the point where it hits the parabola.

This reflective property use can be seen in satellite dishes, telescopic mirrors, car headlights, and search beams, making parabolas useful in real life in many different domains.

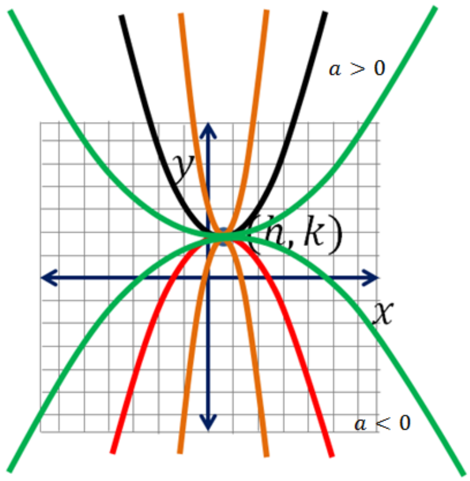


Another way to see parabolas is you were given a quadratic equation in two variables with only one squared term. We can then find the vertex and plot the graph using our basic graphs and the knowledge of transformations.

Equation of a parabola in standard form is given by

Equation of a parabola in standard form: $y = a(x-h)^2 + k$

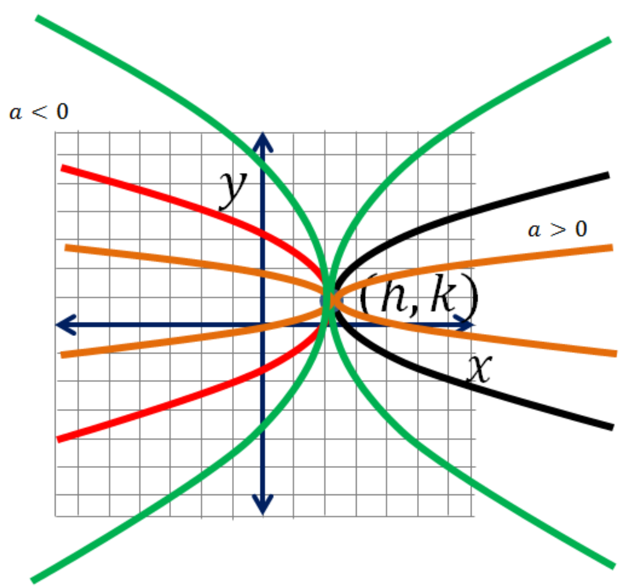
Where (h, k) =Vertex of the parabola, a determines the direction of the parabola (up, or down) and contributes the width of the parabola.



- If $a > 0$, the parabola opens up.
- If $a < 0$, the parabola opens down.
- If $|a| > 1$, the parabola is vertically stretched or is narrower.
- If $|a| < 1$, the parabola is vertically compressed, or is wider.

Equation of a parabola in standard form: $x = a(y-k)^2$ +h

Where (h, k) =Vertex of the parabola and a determines the direction of the parabola and contributes the width of the parabola.



- If $a > 0$, the parabola opens to the right.
- If $a < 0$, the parabola opens to the left.
- If $|a| > 1$, the parabola is narrower.
- If $|a| < 1$, the parabola is wider.

Example

1. Sketch the graph of $y = 5x^2 + 3x + 2$

Solution: We will try to get the equation in the standard form which help in the graphing and finding the vertex. Use completing the square process.

$$y - 2 = 5\left(x^2 + \frac{3}{5}x\right)$$

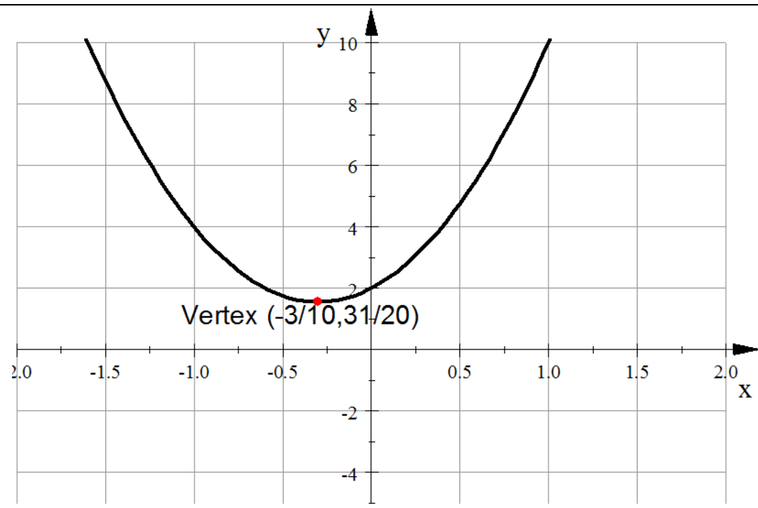
To complete the squares add half of the middle term squared to complete the squares and adjust the other side accordingly.

$$y - 2 + 5\left(\frac{3}{2(5)}\right)^2 = 5\left(x^2 + \frac{3}{5}x + \left(\frac{3}{2(5)}\right)^2\right)$$

$$y - 2 + \frac{9}{20} = 5\left(x + \frac{3}{10}\right)^2$$

Vertex is at $\left(-\frac{3}{10}, 2 - \frac{9}{20}\right) = \left(-\frac{3}{10}, \frac{40}{20} - \frac{9}{20}\right) = \left(-\frac{3}{10}, \frac{31}{20}\right)$

x	y
$-\frac{6}{10}$	2
Vertex $-\frac{3}{10}$	$\frac{31}{20}$
0	2



In general then for any quadratic function in the form $y = ax^2 + bx + c$ we can complete squares and get the equation in the form we looked complete the squares once for this general equations and then use the formula for the vertex.

$y = ax^2 + bx + c$

$y - c = a\left(x^2 + \frac{b}{a}x\right)$ add half of the middle term squared to complete the squares and adjust the other side accordingly.

$$y - c + a\left(\frac{b}{2a}\right)^2 = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right)$$

$$y - c + \frac{b^2}{4a} = \mathrm{a}\left(x + \frac{b}{2a}\right)^2$$

Vertex is at $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right) = \left(-\frac{b}{2a}, -\frac{(b^2-4ac)}{4a}\right)$

If $a > 0$, the value $c - \frac{b^2}{4a}$ is called the minimum value and occurs at $x = -\frac{b}{2a}$.

An example of using this phenomenon is application in business if one wishes to find lowest cost given that the cost function is a quadratic function.

If $a < 0$, the value $c - \frac{b^2}{4a}$ is called the maximum value and occurs at $x = -\frac{b}{2a}$.

An example of using this phenomenon is application in physics if one wishes to find highest point on a trajectory of a projectile when the motion of the particle the object follows a quadratic function.

Equation of a parabola given by $y = ax^2 + bx + c$, with $a \neq 0$, has the vertex (h, k), where $h = -\frac{b}{2a}$ and $k = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + \mathrm{c}$ (basically to find k plug the value of h in for x, in the original equation.

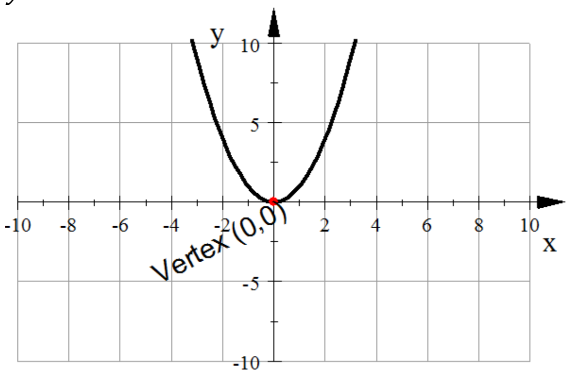
- If $a > 0$, the parabola opens up and the vertex is where the minimum occurs.
- If $a < 0$, the parabola opens to down, and the vertex is where the maximum occurs.
- You can always complete the squares to get the equation in standard form as well.

Equation of a parabola given by $x = ay^2 + by + c$, with $a \neq 0$, has the vertex (h, k), where $k = -\frac{b}{2a}$ and $h = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + \mathrm{c}$ (basically to find h plug the value of k in for y, in the original equation.

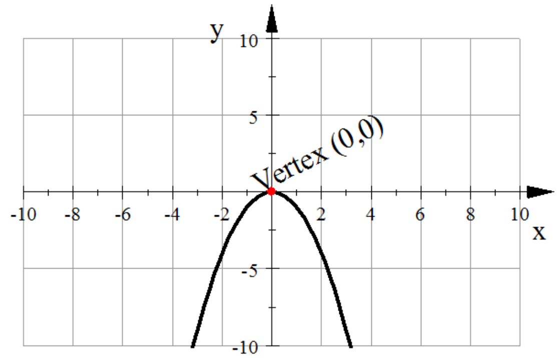
- If $a > 0$, the parabola opens to the right.
- If $a < 0$, the parabola opens to the left.
- You can always complete the squares to get the equation in standard form as well.

Recall the graph of the following relations.

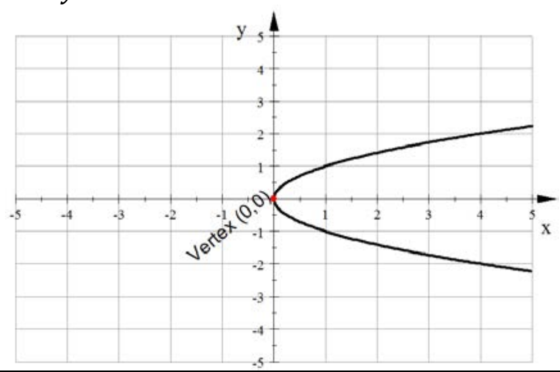
$y = x^2$



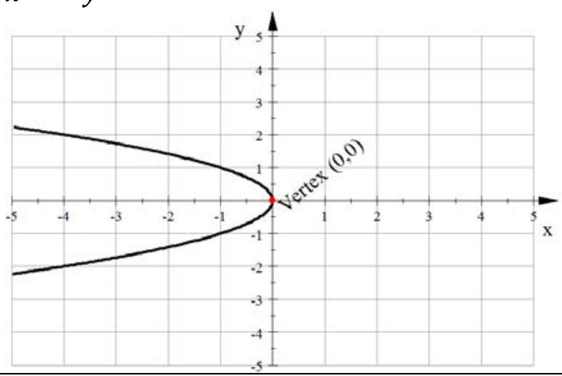
$y = -x^2$



$x = y^2$



$x = -y^2$



Cover Sheet for Sec 2.2a **Name:** ______________________________

Please fill in the chart below.

Terminology	In your own words
1. What is a parabola?	
2. What are all the different forms in which we can write equation of a parabola?	
3. What are the different kinds of parabolas?	
4. What is the maximum value of a function that is quadratic?	
5. What is the minimum value of a quadratic function?	

Any difficulties you encountered in this section?

Video log 2.2a

Sketch the graphs of the following relations. Find the vertex of the parabola and other relevant information if asked.

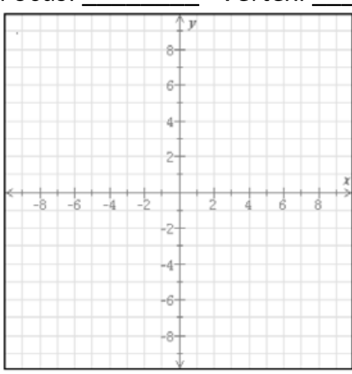
1. $y = 2(x-3)^2 - 5$	2. $y = -2(x-3)^2 - 5$
3. $x = 2(y-3)^2 - 5$	4. $x = -2(y-3)^2 - 5$
5. $y = 3(x+1)^2 + 2$	6. $x = -3(y+1)^2 + 2$
7. $x^2 - 4x + 1 = y$ (Hint: Use completing the squares)	8. $-x^2 - 4x + 1 = y$ (Hint: Use completing the squares)

9. $\frac{2}{3}x^2 - 4x + \frac{1}{3} = y$

10. $-\frac{2}{3}y^2 - 4y - \frac{1}{3} = x$

11. $y = 4x^2$

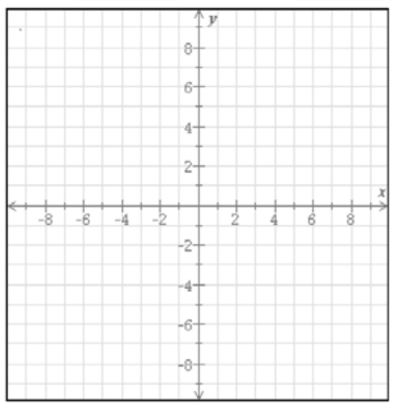
Focus: _________ Vertex: _______



x	y

12. $y + 4 = 4(x-1)^2$

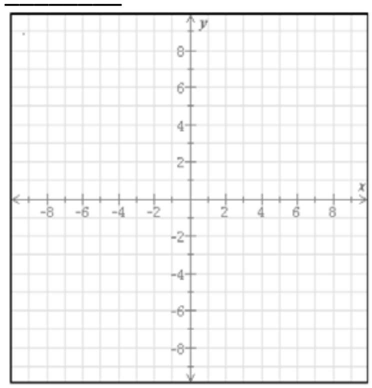
Focus:________ Vertex: _________



x	y

13. $y = 4x^2 - 8x$

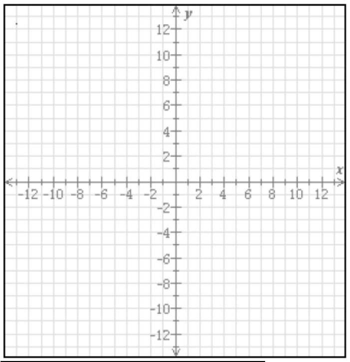
Focus: _________ Vertex: _________



x	y

14. $x^2 + 8x + 3y + 22 = 0$

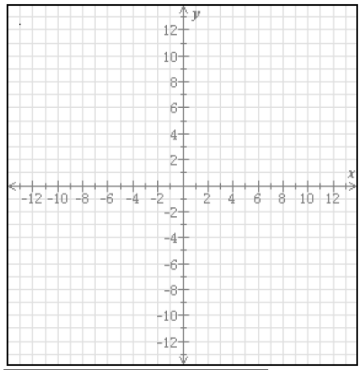
Focus:__________Vertex:_______



x	y

15. $2y^2 - 16y - x + 29 = 0$

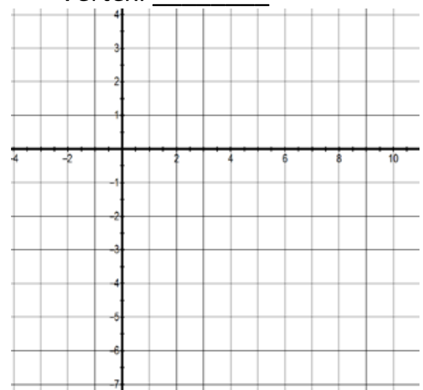
Focus:________ Vertex:__________



x	y

16. $3x^2 - 5x + 4 - 2y = 10$

Vertex: _________



x	y

17. Find the equation of the parabola with vertex $(4,3)$ and focus $(4,6)$.

18. Find the equation of the parabola passing through $(-1,3)$, $(3,3)$ and $(0,2)$. What is the vertex of this parabola?

19. Find the equation of the parabola with vertex $(-4,3)$ and focus $(4,3)$.

2.2b Circles and Ellipses

An ellipse is also characterized by the property that the sum of distances from any point on the ellipse to each of two fixed points (that will be the foci) is constant.

A circle is a special case of an ellipse and can also be characterized as collection of all points that are equidistant from the center and are at fixed distance called the radius. The fixed point is called the center.

General form of the equation of an ellipse is given by
$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, where (h,k) is the center, the axis of symmetries are parallel to the x-axis, and y-axis and their length's are $2a$, $2b$ respectively. The larger length is referred to as major axis and the shorter as the minor axis. The foci are located on the major axis a distance $c=\sqrt{|a^2-b^2|}$ from the center.

When $a=b$ we have a circle of radius a and center (h,k).

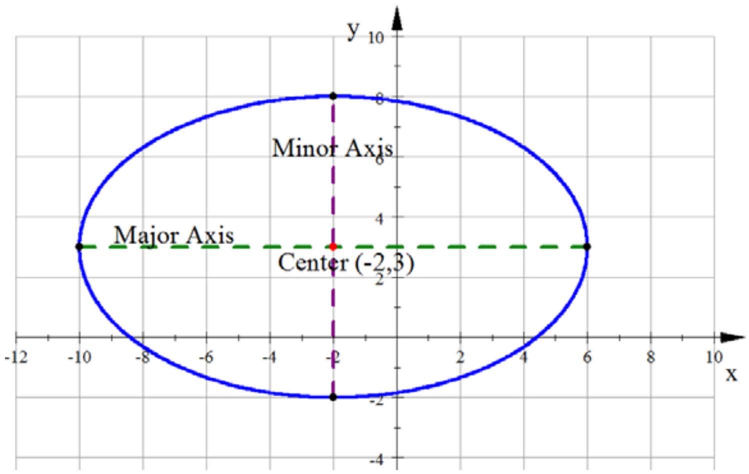
Reflective Property of Ellipse: One of the properties of an ellipse is that if an energy source located at one of foci, and the ellipse is made of a reflective surface the energy bounces of the ellipse and merges onto the other focus point.

This property is used in whispering galleries and even in medical equipment like the lithotripter which helps target underwater shockwaves to break up a kidney stone.

Example

1. Sketch the graph of $\frac{(x+2)^2}{64}+\frac{(y-3)^2}{25}=1$. Find the center, major and minor axis, and the foci.

Solution: Center is $(-2,3)$. $a=8$, $b=5$, $c=\sqrt{|64-25|}=\sqrt{39}$ and major axis is of length 16 and minor axis of length 10. Foci are located at $\left(-2+\sqrt{39},3\right),\left(-2-\sqrt{39},3\right)$



2. Sketch the graph of the ellipse and find the center of the ellipse.

$4x^2+16x+9y^2+18y=119$

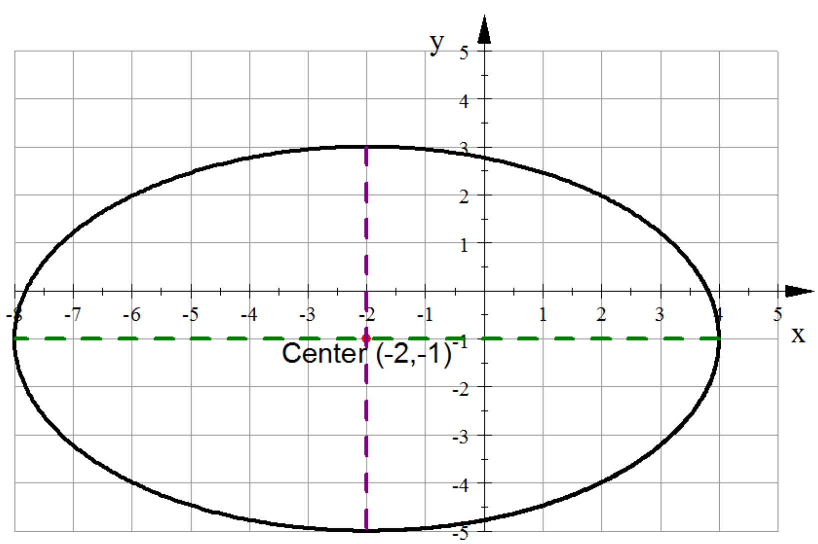
Solution:

$4(x^2+4x)+9(y^2+2y)=119$

$4(x+2)^2+9(y+1)^2=119+16+9$

$$\frac{(x+2)^2}{36}+\frac{(y+1)^2}{16}=1$$

Center is $(-2,1)$, $a=6, b=4$.

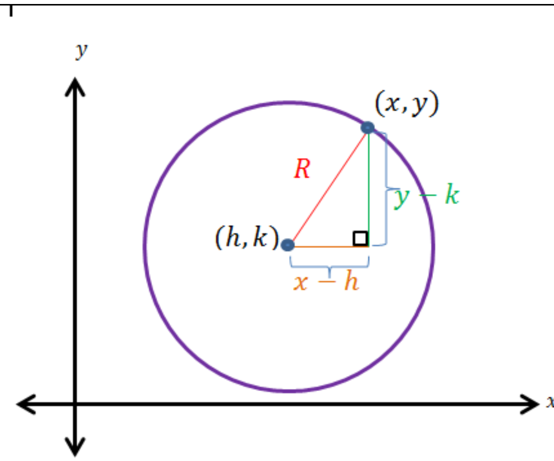
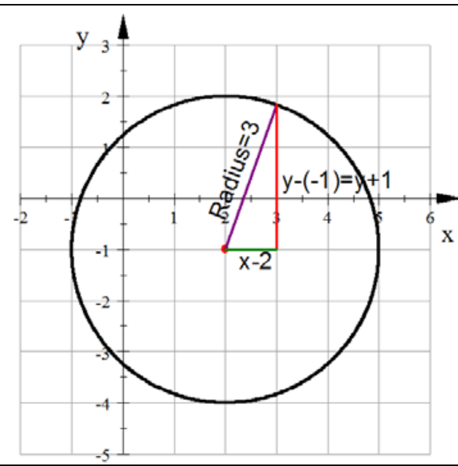


Circles

Equations of Circles

http://www.youtube.com/watch?v=fzNXmoCHRCk (10 mins)

We can derive equation of a circle given the center and radius using the distance formula.

Hint: The distance between two points (x_1, y_1), and (x_2, y_2) is given by $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ or $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. That would suggest the equation in example 6 on the previous page represents a circle centered at $(0,0)$ with radius of 4. So it is clear that in general then, the equation $(x - h)^2 + (y - k)^2 = R^2$ would represent a circle with center (h, k) and radius R.	
Example 1. Sketch the graph of $(x - 2)^2 + (y + 1)^2 = 9$ Solution: So the graph of the equation $(x - 2)^2 + (y + 1)^2 = 3^2$ would be a circle with radius of 3 and center of $(2, -1)$ since $(x - 2)^2 + (y + 1)^2 = 3^2$ can be written as $(x - 2)^2 + (y - (-1))^2 = 3^2$	

Equation of a Circle in Standard Form

In general an equation of a circle in the form $(x - h)^2 + (y - k)^2 = r^2$, where the center is at (h, k) and the radius is R is said to be in **standard form**.

Examples

1. Find equation of a circle with
   a. Center at (3,5) and radius of 2.

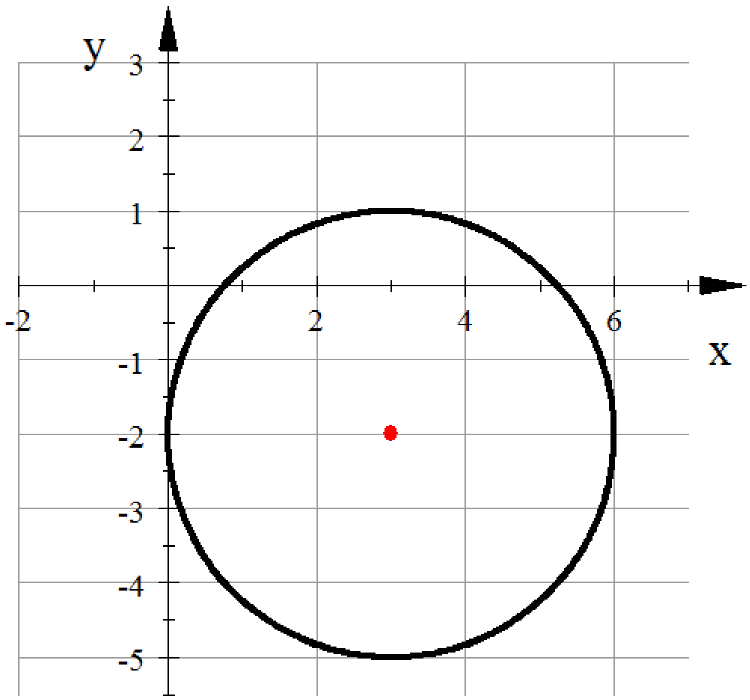
   Solution: An equation for the circle centered at $(3, 5)$ with radius 2 is

   $(x - 3)^2 + (y - 5)^2 = 2^2$

   b. Center $(-3,7)$ and radius of 5.
   Solution: The circle of radius 5 centered at $(-3, 7)$ is represented by

$\left(x-(-3)\right)^2+(y-7)^2=5^2$ or $(x+3)^2+(y-7)^2=25$.

2. Find the center and radius of the circle given by $(x-6)^2+(y+3)^2=49$
   Solution: The equation $(x-6)^2+(y+3)^2=49$ represents a circle centered at $(6,-3)$ of radius 7.
3. Find the equation of the circle whose graph is given below.



Solution: The equation for the circle graphed to the right is obtained by reading off the center as $(3,-2)$ and radius as $r=3$ from the graph. Thus the equation is $(x-3)^2+(y+2)^2=9$.

Cover Sheet for Sec 2.2b **Name: ______________________________**

Please fill in the chart below.

Terminology	In your own words
1. What is a circle?	
2. What is an ellipse? How is it different than a circle?	
3. What is a major axis and a minor axis?	
4. How do we find the center and the foci of an ellipse?	
5. How do we find center and radius of a circle.	

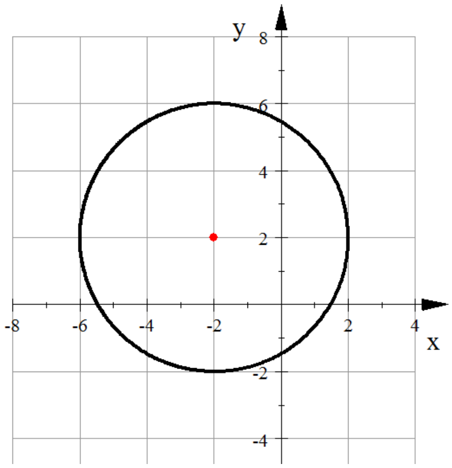
Any difficulties you encountered in this section?

Video Log 2.2b

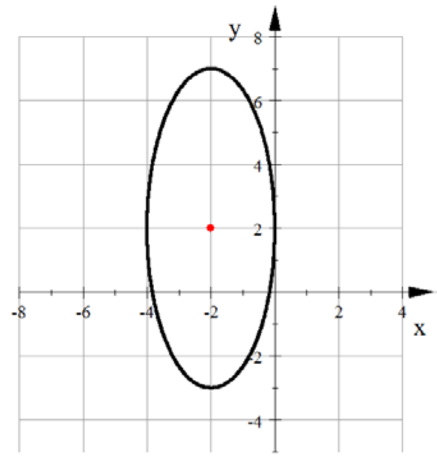
Sketch the graph of the following. If the curve is an ellipse find the center, major and minor axis, foci. If the curve is a circle, find the center and radius.

1. $x^2 + y^2 = 36$	2. $(x-3)^2 + (y-4)^2 = 25$
3. $(x+2)^2 + (y-1)^2 = 9$	4. $4x^2 + 8x + 4y^2 - 24y = 24$
5. Sketch graphs and write equations that represent each circle described below.	
a. The center is at $(2, 7)$ and the radius is $r = 9$.	b. Two endpoints of a diameter are the points $(1, 2)$ and $(5,2)$.

6. Write the equation in standard form for the graph of the circle shown below.



7. Write equation of the ellipse in standard form for the graph of the ellipse shown below.

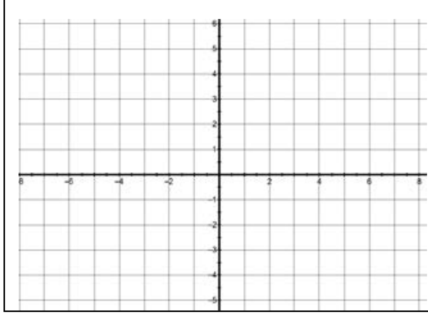


8. Find equation of the circle whose diameter has endpoints $(-1,4)$, and $(4,-2)$.

9. Find equation of the circle whose diameter has endpoints $(2,-1)$, and $(3,2)$.

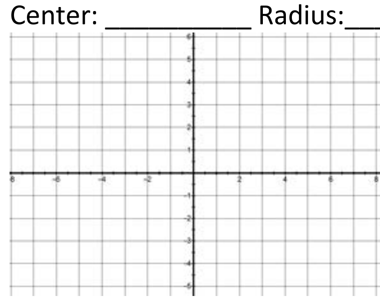
10. $(x+5)^2 + (y-3)^2 = 9$

Center: _________ Radius:_______



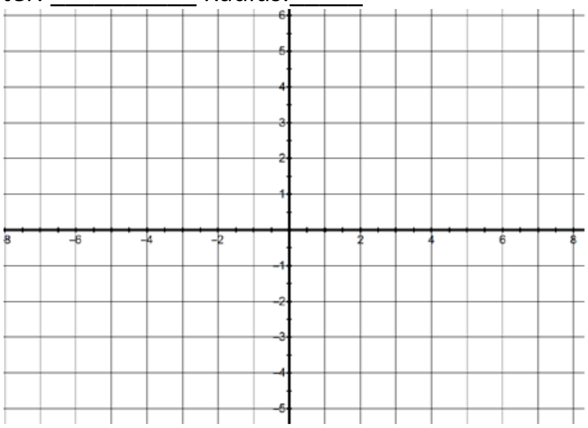
11. $(x-2)^2 + (y+1)^2 = \frac{25}{4}$

Center: ___________ Radius:______



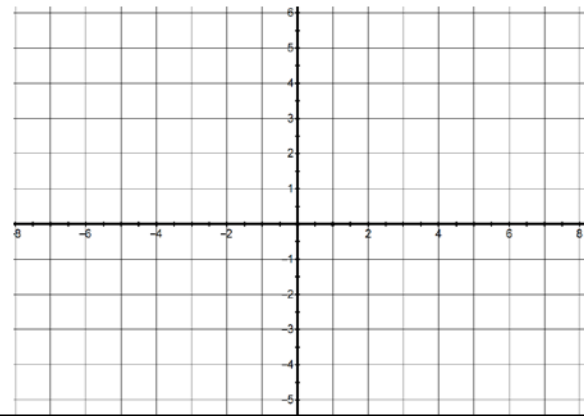
12. $x^2 + 4x + y^2 - 6y = 12$

Center: ___________ Radius:______



13. $2x^2 + 8x + 2y^2 - 14y = -\frac{29}{2}$

Center: _________ Radius:_______



14. Find equation of the circle with center $(-3,5)$ and radius 5.

15. Find the equation of a circle with center $(2,5)$, and passing through $(-2,2)$.

16. Find equation of the circle whose diameter has endpoints $(-1,4)$, and $(4,-2)$.

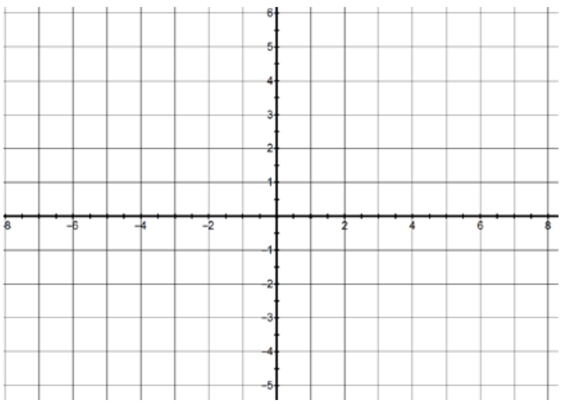
17. Find equation of the circle whose diameter has endpoints $(-1,4)$, and $(4,-2)$.

18. $\frac{(x+3)^2}{16} + \frac{(y-1)^2}{25} = 1$

☐ Ellipse	☐ Hyperbola
☐ Circle	☐ Parabola

Center: ___________

$a =$_________ $b =$_________

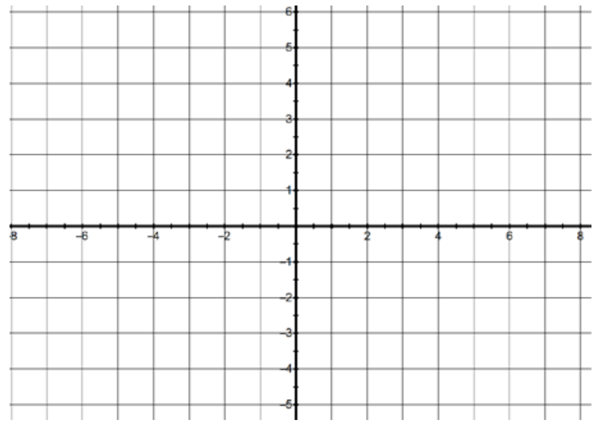


19. $16x^2 + y^2 - 16 = 0$

☐ Ellipse	☐ Hyperbola
☐ Circle	☐ Parabola

Center: ___________

$a =$_________ $b =$_________

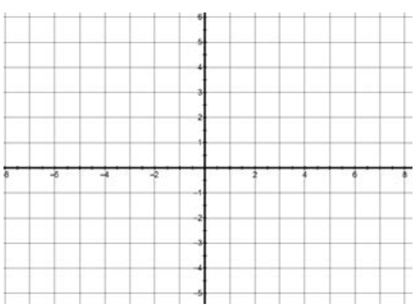


20. $4x^2 - 16x + 9y^2 - 18y = 11$

☐ Ellipse	
☐ Circle	☐ Parabola

Center: ___________

$a =$_________ $b =$_________

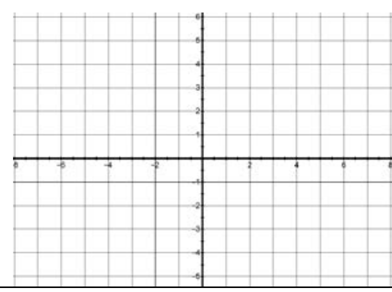


21. $\frac{(x+3)^2}{16} + \frac{(y-1)^2}{25} = 1$

☐ Ellipse	
☐ Circle	☐ Parabola

Center: ___________

$a =$_________ $b =$_________



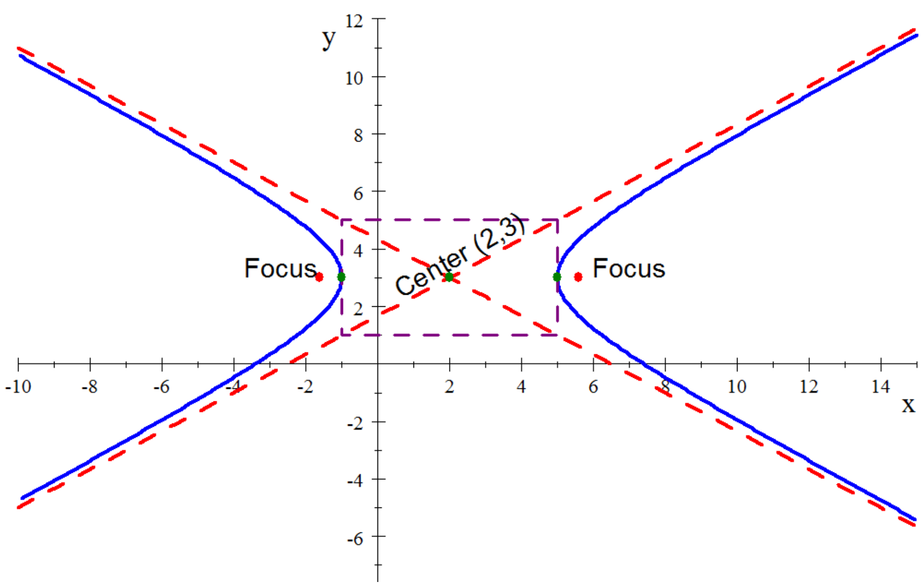
2.2c Hyperbolas

A hyperbola is characterized by the property that the difference of the distances from any point on the hyperbola to each of the two fixed points (that will be the foci) is constant.
General form of the equation of the hyperbola is given by
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, or $-\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, where (h, k) is the center, the asymptotes have slopes of $\pm\frac{b}{a}$. The foci are located a distance of $c = \sqrt{a^2 + b^2}$ from the center.

Find the reflective property of hyperbolas and for extra credit find one application of the reflective property.

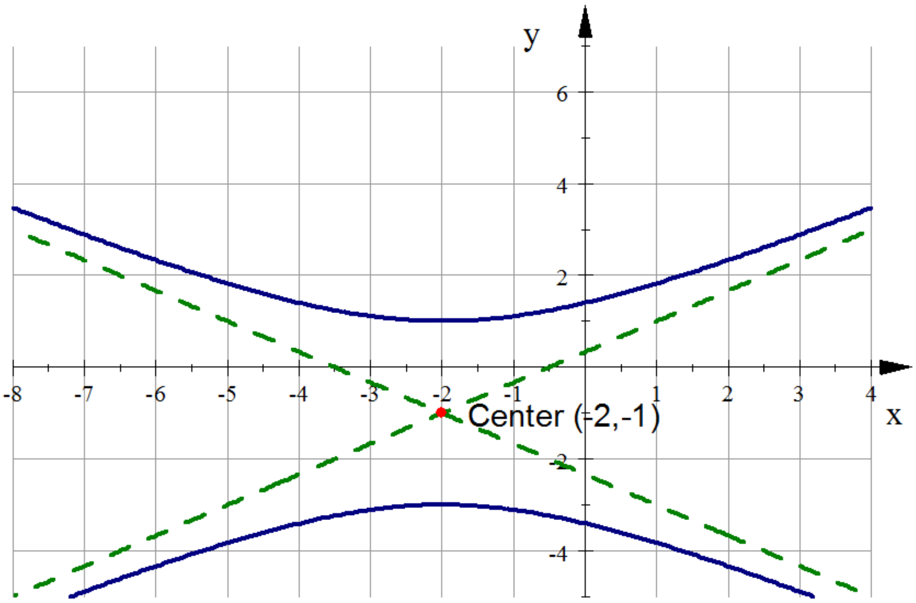
Example

1. Sketch the graph of $\frac{(x-2)^2}{9} - \frac{(y-3)^2}{4} = 1$. Find the center, the foci and the asymptotes.
   Solution: Center is $(2,3)$, $a = 3$, $b = 2$, $c = \sqrt{9+4} = \sqrt{13}$, foci are located at $(2 \pm \sqrt{13}, 3)$.
   Asymptotes are given by . Asymptotes are given by $y - 3 = \pm\frac{2}{3}(x-2)$



2. Plot the graph of hyperbola and find the center and the asymptote.
   $-4x^2 - 16x + 9y^2 + 18y = 43$

Solution:
$-4(x^2 + 4x) + 9(y^2 + 2y)$ =43
$-4(x+2)^2 + 9(y+1)^2 = 43 - 16 + 9$
$-4(x+2)^2 + 9(y+1)^2 = 36$
$$-\frac{(x+2)^2}{9} + \frac{(y+1)^2}{4} = 1$$
Center is $(-2,1)$, $a = 3, b = 2$, asymptotes are given by $y + 1 = \pm\frac{2}{3}(x+2)$


y
6
4
2
-2
-4
-8
-7
-6
-5
-4
-3
-2
-1
1
2
3
4
x
Center (-2,-1)

Cover Sheet for Sec 2.2c **Name:** ______________________________

Please fill in the chart below.

Terminology	In your own words
1. What is a hyperbola?	
2. What are the different kinds of hyperbolas?	
3. How do you find center and foci of a hyperbola?	
4. How do we find the asymptotes of a hyperbola?	

Any difficulties you encountered in this section?

Video log 2.2c

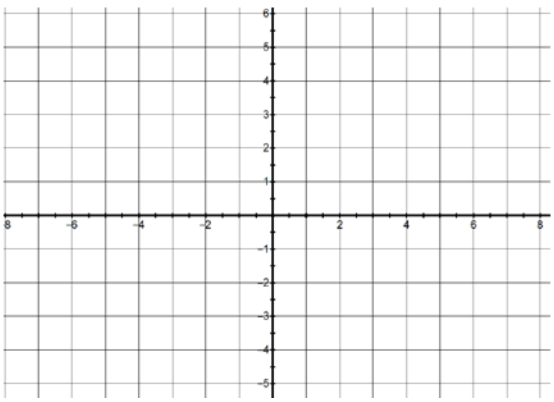
Where asked sketch the graph and label all the relevant points and asymptotes if any. Find any other information asked in the specific problems.

22. $4x^2 - 16x - 9y^2 + 18y = 29$

☐ Ellipse	☐ Hyperbola
☐ Circle	☐ Parabola

Center: ___________

$a =$_________ $b =$_________

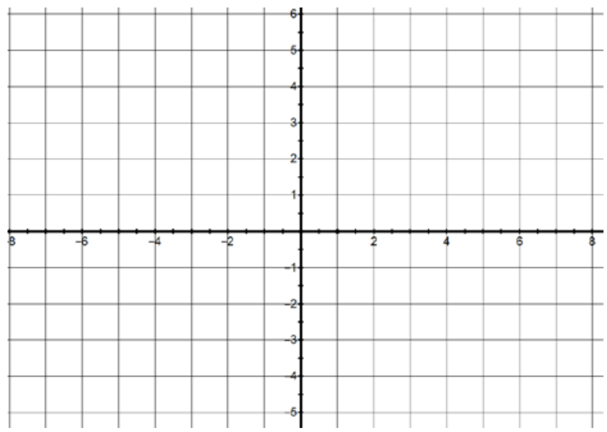


23. $\frac{(x+3)^2}{16} - \frac{(y-1)^2}{25} = 1$

☐ Ellipse	☐ Hyperbola
☐ Circle	☐ Parabola

Center: ___________

$a =$_________ $b =$_________

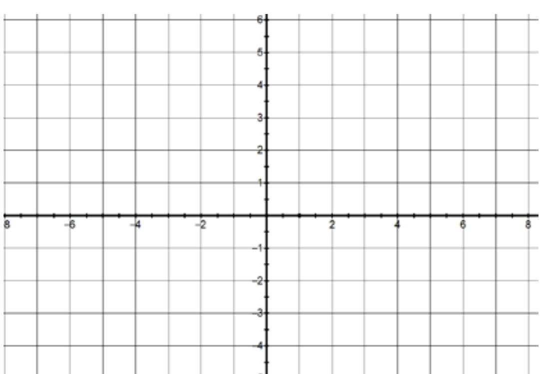


24. $16x^2 - y^2 - 16 = 0$

☐ Ellipse	☐ Hyperbola
☐ Circle	☐ Parabola

Center: ___________

$a =$_________ $b =$_________

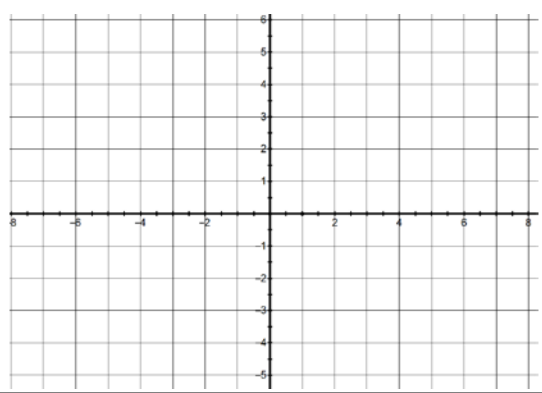


25. $4x^2 - 16x - 9y^2 + 18y = 29$

☐ Ellipse	☐ Hyperbola
☐ Circle	☐ Parabola

Center: ___________

$a =$_________ $b =$_________

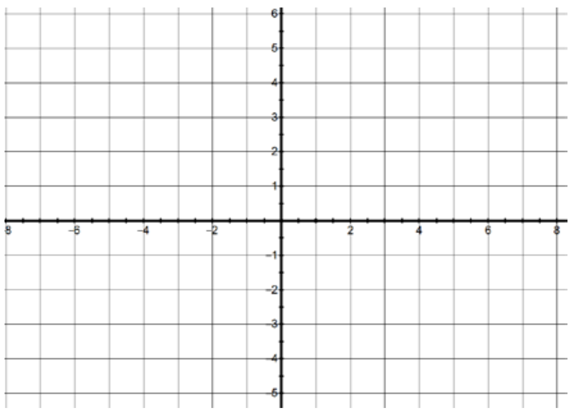


26. $\frac{(y+3)^2}{16} - \frac{(x-1)^2}{25} = 1$

☐ Ellipse	☐ Hyperbola
☐ Circle	☐ Parabola

Center: ___________

$a =$_________ $b =$_________

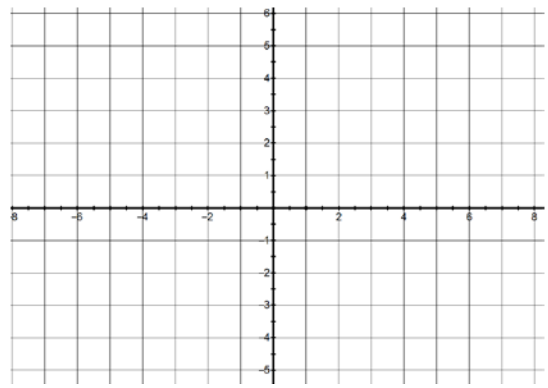


27. $-4x^2 - 16x + 9y^2 + 18y = 43$

☐ Ellipse	☐ Hyperbola
☐ Circle	☐ Parabola

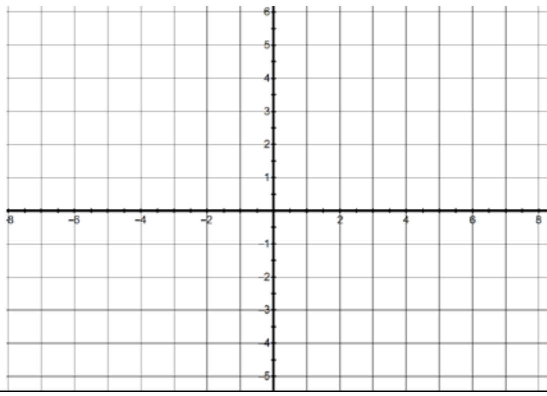
Center: ___________

$a =$_________ $b =$_________



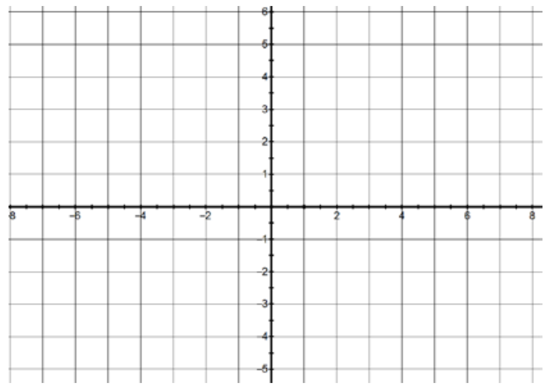
28. $y^2 = 2x - x^2$

☐ Ellipse	☐ Hyperbola
☐ Circle	☐ Parabola



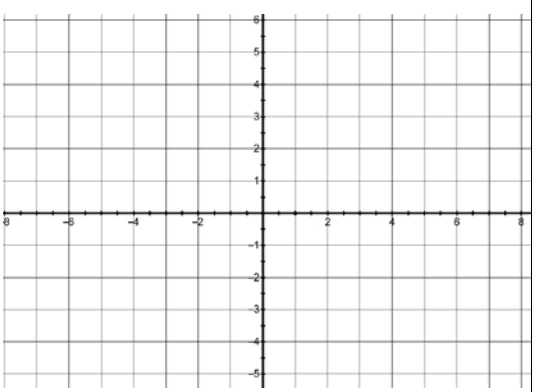
29. $x^2 + y^2 - 2x + 4y = 4$

☐ Ellipse	☐ Hyperbola
☐ Circle	☐ Parabola



30. $3y^2 - 4x + 6y = 4$

☐ Ellipse	☐ Hyperbola
☐ Circle	☐ Parabola



31. $3y^2 + 4x + 6y = 4$

☐ Ellipse	☐ Hyperbola
☐ Circle	☐ Parabola