1.4 Arithmetic of Functions

In the previous sections we saw types of functions. As a mathematician any time we get a new object to play with which in this case is functions, we want to see if we can do arithmetic with it.

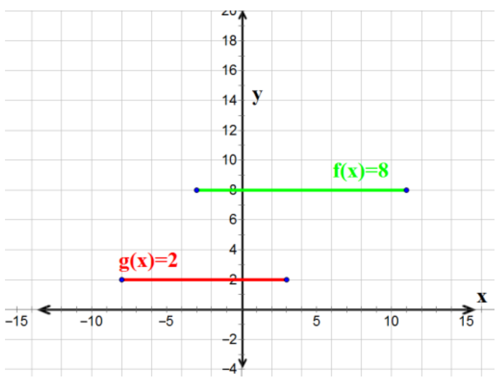
Playing

Below are some functions and we will see what happens when we do arithmetic with it. Attempt the problems using your instinct before looking at answers on the next page.

Practice Examples

1. Evaluate the following

A. $f(x) = 8, for - 3 \leq x \leq 11,$
$g(x) = 2, for - 8 \leq x \leq 3$



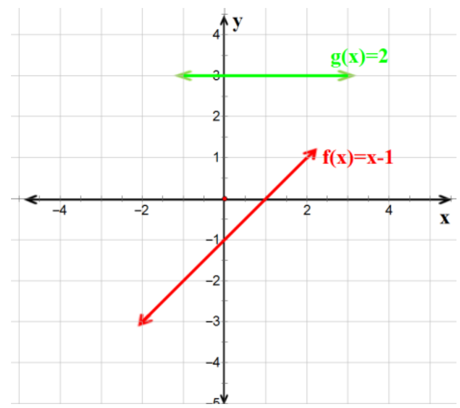
I. $f(x) + g(x) =$

II. $f(x) - g(x) =$

III. $f(x) \times g(x) =$

IV. $f(x) \div g(x)$ or $\frac{f(x)}{g(x)} =$

B. $f(x) = x - 1$, for all real numbers x
$g(x) = 3$, for all real numbers x



I. $f(x) + g(x) =$

II. $f(x) - g(x) =$

III. $f(x) \times g(x) =$

IV. $g(x) \div f(x)$ or $\frac{g(x)}{f(x)} =$

V. $f(4) + g(4) =$

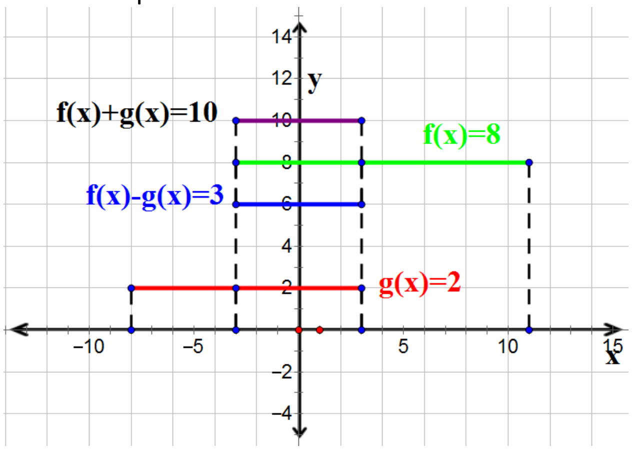
Please attempt before looking at the solutions ahead.

Practice Examples Solutions

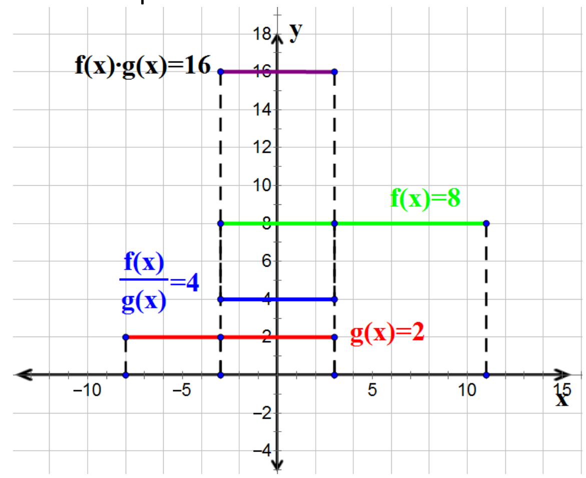
1. Evaluate the following

   A. $f(x) = 8, for - 3 \leq x \leq 11,$
   $g(x) = 2, for - 8 \leq x \leq 3$

Picture for parts I. and II.



Picture for parts III. and IV.



You can see from the pictures above that adding, subtracting, multiplying or dividing two functions together is not even possible unless we restrict our domains for both functions $f(x)$ and $g(x)$ to the interval $-3 \leq x \leq 3$ (which is the intersection of the two domains). This is the interval where both functions are defined and we can do arithmetic with the values of the two given function. So domain of all the functions below is $[-3,3]$.

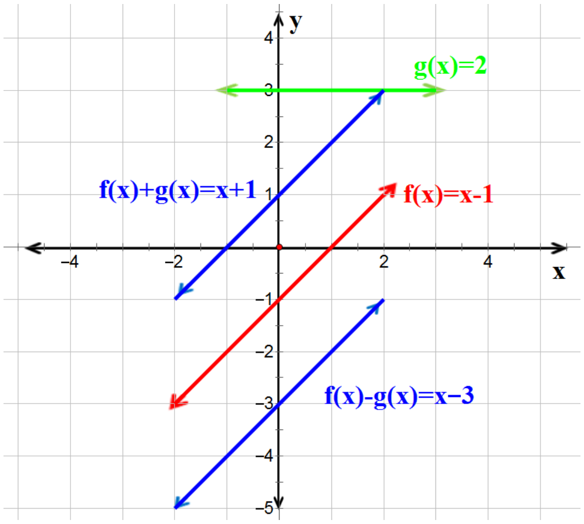
I. $f(x) + g(x) = 8 + 2 = 10$, for $-3 \leq x \leq 3$

II. $f(x) - g(x) = 8 - 2 = 6$, for $-3 \leq x \leq 3$

III. $f(x) \times g(x) = 8 \times 2 = 16$, for $-3 \leq x \leq 3$

IV. $f(x) \div g(x)$ or $\frac{f(x)}{g(x)} = \frac{8}{2} = 4$, for $-3 \leq x \leq 3$

B. $f(x) = x - 1$, for all real numbers x
$g(x) = 2$, for all real numbers x



I. $f(x) + g(x) = x - 1 + 2 = x + 1$ for all real numbers (note that, the effect of adding a 2 to the function $x - 1$ is all the points on the line $y = x - 1$ shifted up 2 units)
Domain here is all real numbers

II. $f(x) - g(x) = x - 1 - 2 = x - 3$ for all real numbers (note that, the effect of adding a 2 to the function $x - 1$ is all the points on the line $y = x - 1$ shifted up 2 units)
Domain here is all real numbers

III. $f(x) \times g(x) = (x - 1)2 = 2x - 2$
Domain here is all real numbers

IV. $g(x) \div f(x)$ or $\frac{g(x)}{f(x)} = \frac{2}{x-1}$, domain is all real numbers not equal to 1. Since when $x = 1$ we get zero in the denominator. So we can see that division of functions is only possible as long as the denominator remains non-zero.
Domain is $(-\infty, 1) \cup (1, \infty)$

V. $f(4) + g(4) = 4 - 1 - 2 = 1$

As you can see adding, subtracting, multiplying or dividing two functions creates new functions. We use the following notation for them

$$f(x) + g(x) = (f + g)(x)$$
$$f(x) - g(x) = (f - g)(x)$$
$$f(x) \times g(x) = (f \cdot g)(x)$$
$$\frac{f(x)}{g(x)} = \left(\frac{f}{g}\right)(x)$$

The notation of $f+g, f-g, f\cdot g, \frac{f}{g}$ are the names of the function that we get by doing the algebraic notation involved in the notation. Remember the domain of these new functions is the intersection of the individual functions involved in the operations. In case of division we need to extra precaution to make sure the denominator is non-zero and so occasionally we may have to also avoid all numbers that make denominators zeros from the domain as well.

Video log 1.4a

1 $f(x)=3x-1$ and $g(x)=x^2+2$ i. $(f+g)(x)=$ ______________ ii. Domain of $(f+g)$ iii. $(f+g)(3)=$ ______________ iv. $(fg)(x)=$ ______________ v. Domain of $(fg)=$ ___________ vi. $(fg)(0)=$ ______________	2 $f(x)=5x^2+1$ and $g(x)=x-2$ i. $\left(\frac{f}{g}\right)(x)=$ ______________ ii. Domain of $\left(\frac{f}{g}\right)$ iii. $\left(\frac{f}{g}\right)(3)=$ ______________ iv. $(f-g)(x)=$ ______________ v. Domain of $(f-g)=$ ___________ vi. $(f-g)(0)=$ ______________
3. If $f(x)=(2+x)(-4+x)$ and $g(x)=(1-x)(1+x)$. Find all values that are NOT in the domain of $\frac{f}{g}$. If there are more than one value separate them with commas.	
4. EXTRA CREDIT: If $f(x)=\sqrt{x-1}$ and $g(x)=\ln(x-3)$. Evaluate the following A. Domain of $f+g$ B. Domain of $\frac{f}{g}$	

Composition of Functions

Composition of functions Part 1 https://www.youtube.com/watch?v=q3KcPmQfACo

Composition of functions Part 2 https://www.youtube.com/watch?v=HnKUOLC_PKs

There is one more operation we may want to consider called composition of functions which involves taking function of a function if it is valid.

Playing

If we let the function $f(x) = x - 1$ and $g(x) = \sqrt{x}$ then we can evaluate
$f(g(x)) = f(\sqrt{x}) = \sqrt{x} - 1$. This is possible since remember we said evaluating function $f(\quad)$ means replacing whatever goes in for the blank in place of x.

Clearly that new output is a function that is different from the original function $f(x)$. We also would have to make sure that the input in f is valid.

<u>Composition of Functions</u> : Given two functions $f(x)$ and $g(x)$, a third function called the composite function with notation $f \circ g(x)$ is a function defined by evaluating $f(g(x))$ for appropriate values of x, or you can think of it as $x \xrightarrow{g} g(x) \xrightarrow{f} f(g(x))$.
Domain of $(f \circ g)(x)$: $\{x | x$ is in the domain of $g(x)$ for which $g(x)$ is in the domain of $f(x)\}$

So you can think about $f \circ g(x)$ as evaluating function of a function starting with the input of x, getting an output from computing $g(x)$, this output then becomes the input in the function f and the output $f(g(x))$ is the answer to the composite function $f \circ g(x)$.

Only those values of domain of g are allowed in the domain of $f \circ g(x)$ function for which $g(x)$ values are in domain of $f(x)$.

Practice Examples

1. Evaluate the composite functions below. Find the domain of the composite functions.

a. Let $f(x) = 2x - 1$ and $g(x) = \sqrt{x}$. Find i. $(f \circ g)(x)$ ii. $(g \circ f)(x)$ iii. $Domain\ of\ (f \circ g)(x)$ iv. $Domain\ of\ (g \circ f)(x)$	b. Let $f(x) = \frac{2}{x-1}$ and $g(x) = x + 5$. Find i. $(f \circ g)(x)$ ii. $(g \circ f)(5)$ iii. $Domain\ of\ (f \circ g)(x)$

c. Let $f(x) = 2x - 1$ and $g(x) = \frac{x+1}{2}$. Find	d. Let $f(x) = \frac{1}{x-1}$ and $g(x) = \frac{1}{x} + 1$. Find
i. $(f \circ g)(3)$	i. $(f \circ g)(6)$
ii. $(g \circ f)(5)$	ii. $(g \circ f)(2)$
iii. $(f \circ g)(x)$	iii. $(f \circ g)(x)$
iv. $(g \circ f)(x)$	iv. $(g \circ f)(x)$
v. $f^{-1}(x)$	v. $f^{-1}(x)$
vi. $g^{-1}(x)$	vi. $g^{-1}(x)$
After solving problems 1c and 1d, do you notice any peculiar behaviors? Please note your observations below.	

Practice Examples Solutions

1. Evaluate the composite functions below. Find the domain of the composite functions.

a. Let $f(x) = 2x - 1$ and $g(x) = \sqrt{x}$. Find

i. $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = 2\sqrt{x} - 1$ (see below for alternate explanation)

$$x \xrightarrow{g} \sqrt{x} \xrightarrow{f} 2\sqrt{x} - 1$$

Domain:$[0,\infty)$

ii. $(g \circ f)(x) = g(f(x)) = g(2x - 1) = \sqrt{2x - 1}$

$$x \xrightarrow{f} 2x - 1 \xrightarrow{g} \sqrt{2x - 1}$$

Domain:$\left[\frac{1}{2}, \infty\right)$ since otherwise the end result would not make sense.

iii. Domain of $(f \circ g)(x)$ =Domain:$[0,\infty)$

iv. Domain of $(g \circ f)(x) = \left[\frac{1}{2}, \infty\right)$

b. Let $f(x) = \frac{2}{x-1}$ and $g(x) = x + 5$. Find

i. $(f \circ g)(x) = f(g(x)) = f(x + 5) = \frac{2}{x+5-1} = \frac{2}{x+4}$ (see below for alternate explanation)
Here $x \neq -4$.

$$x \xrightarrow{g} x + 5 \xrightarrow{f} \frac{2}{x + 5 - 1} = \frac{2}{x + 4}$$

ii. $(g \circ f)(5) = g(f(5)) = g\left(\frac{2}{5-1}\right) = g\left(\frac{2}{4}\right) = g\left(\frac{1}{2}\right) = 5 + \frac{1}{2} = 5\frac{1}{2}$

$$x \xrightarrow{f} \frac{2}{5 - 1} = \frac{2}{4} = \frac{1}{2} \xrightarrow{g} \frac{1}{2} + 5$$

iii. Domain of $(f \circ g)(x)$
$= (-\infty, -4) \cup (-4, \infty)$

Since denominator cannot be zero.

c. Let $f(x) = 2x - 1$ and $g(x) = \frac{x+1}{2}$. Find

i. $(f \circ g)(3) = f(g(3)) = f\left(\frac{3+1}{2}\right) = f(2) = 2(2) - 1 = 4 - 1 = 3$

$$3 \xrightarrow{g} \frac{3+1}{2} = \frac{4}{2} = 2 \xrightarrow{f} 2(2) - 1 = 4 - 1 = 3$$

Notice input was 3 and output is 3 also.

ii. $(g \circ f)(5) = g(f(5)) = g(10-1) = g(9) = \frac{9+1}{2} = 5$

$$5 \xrightarrow{f} 2(5) - 1 = 10 - 1 = 9 \xrightarrow{g} \frac{9+1}{2} = \frac{10}{2} = 5$$

Again notice input was 5 and output was 5.

iii. $(f \circ g)(x) = f\left(\frac{x+1}{2}\right) = 2\left(\frac{x+1}{2}\right) - 1 = x$

iv. $(g \circ f)(x) = g(2x-1) = \frac{2x-1+1}{2} = x$

v. $f^{-1}(x)$

$$y = 2x - 1$$
$$x = 2y - 1$$
$$\frac{x+1}{2} = y$$
$$f^{-1}(x) = \frac{x+1}{2}$$

vi. $g^{-1}(x)$

$$y = \frac{x+1}{2}$$
$$x = \frac{y+1}{2}$$
$$2x = y + 1 \text{ or } 2x - 1 = y$$
$$g^{-1}(x) = 2x - 1$$

d. Let $f(x) = \frac{1}{x-1}$ and $g(x) = \frac{1}{x} + 1$. Find

i. $(f \circ g)(6) = f(g(6)) = f\left(\frac{1}{6} + 1\right) = \frac{1}{\frac{1}{6}+1-1} = \frac{1}{\frac{1}{6}} = 6$

You can see that f is undoing what g did to 6.

ii. $(g \circ f)(2) = g(f(2)) = g\left(\frac{1}{2-1}\right) = g(1) = \frac{1}{1} + 1 = 2$

You can see that g is undoing what f did to 2.

iii. $(f \circ g)(x) = f\left(\frac{1}{x} + 1\right) = \frac{1}{\frac{1}{x}+1-1}$

$$= \frac{1}{\frac{1}{x}} = x$$

iv. $(g \circ f)(x) = g\left(\frac{1}{x-1}\right) = \frac{1}{\frac{1}{x-1}} + 1$

$$= x - 1 + 1 = x$$

v. $f^{-1}(x)$

$y = \frac{1}{x-1}$ or $x = \frac{1}{y-1}$

$y - 1 = \frac{1}{x}$ or $y = \frac{1}{x} + 1$

$$f^{-1}(x) = \frac{1}{x} + 1$$

vi. $g^{-1}(x)$

$y = \frac{1}{x} + 1$ or $x = \frac{1}{y} + 1$

$x - 1 = \frac{1}{y}$ or $y = \frac{1}{x-1}$

$$g^{-1}(x) = \frac{1}{x-1}$$

After solving problems 1c and 1d, do you notice any peculiar behaviors? Please note your observations below.

We see that $(f \circ f^{-1})(x) = x$ and that $(f^{-1} \circ f)(x) = x$

For one-to-one function $y = f(x)$, $(f \circ f^{-1})(x) = x$ and that $(f^{-1} \circ f)(x) = x$.

Video log 1.4b Cover Sheet

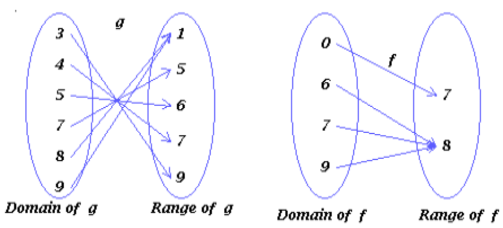
Name:________________________________

Summary

Concept	In your words describe what these concepts mean to you
What is a composite function?	
What is the relationship of a function, it's inverse and composite of these two functions?	
How do you find domain and range of a composite function? Explain.	

Please list any problems you were stuck on or concepts that you would like more help on in class.

Video log 1.4b

<table>
<tr><td colspan="4">1. Two functions g and f are defined in the figure below. Find the domain and range of the compositions $f{\circ}g(x) = f(g(x))$, and $g{\circ}f(x) = g(f(x))$. Then evaluate the function values below.</td></tr>
<tr><td colspan="2">
</td><td>Domain of f

Domain of g

Domain of $(f \circ g)(x) = f(g(x))$

Domain of $(g \circ f)(x) = g(f(x))$</td><td>Range of f

Range of g

Range of $(f \circ g)(x) = f(g(x))$

Range of $(g \circ f)(x) = g(f(x))$</td></tr>
<tr><td>a. $f \circ g(3)$</td><td>b. $g \circ f(9)$</td><td>c. $f \circ g(7)$</td><td>d. $g \circ f(0)$</td></tr>
</table>

<table>
<tr><td colspan="4">2. For the real valued functions $g(x) = \frac{x+6}{x-5}$, and $f(x) = 2x - 7$ find the compositions listed below and specify domains of these functions using interval notation. Then evaluate the values of the function listed below.</td></tr>
<tr><td colspan="2">$(f \circ g)(x) = f\big(g(x)\big) =$</td><td colspan="2">$(g \circ f)(x) = g\big(f(x)\big) =$</td></tr>
<tr><td colspan="2">Domain of f

Domain of g</td><td colspan="2">Range of f

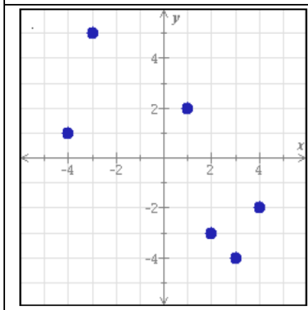
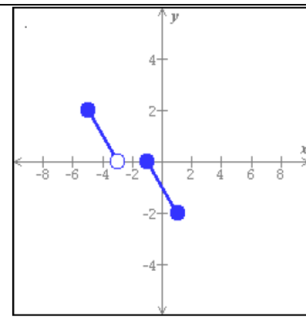
Range of g</td></tr>
<tr><td colspan="2">Domain of $(f{\circ}g)(x) = f(g(x))$

Domain of $(g{\circ}f)(x) = g(f(x))$</td><td colspan="2">Range of $(f{\circ}g)(x) = f(g(x))$

Range of $(g{\circ}f)(x) = g(f(x))$</td></tr>
<tr><td>a. $(f{\circ}g)(x)$</td><td>b. $(g{\circ}f)(6)$</td><td>c. $(f{\circ}g)(6)$</td><td>d. $(g{\circ}f)(0)$</td></tr>
</table>

3. For the real valued functions $g(x) = \sqrt{x+8}$, and $f(x) = x^2 + 7$ find the compositions listed below and specify domains of these functions using interval notation. Then evaluate the values of the function listed below.			
$(f \circ g)(x) = f(g(x)) =$		$(g \circ f)(x) = g(f(x)) =$	
Domain of f Domain of g		Range of f Range of g	
Domain of $(f \circ g)(x) = f(g(x))$ Domain of $(g \circ f)(x) = g(f(x))$		Range of $(f \circ g)(x) = f(g(x))$ Range of $(g \circ f)(x) = g(f(x))$	
e. $f \circ g(x)$	f. $g \circ f(1)$	g. $g \circ f(x)$	h. $f \circ g(1)$

4. For each of the pairs of functions below find $f(g(x))$ and $g(f(x))$. Then determine whether f and g are inverses of each other. Simplify your answers as much as possible. (Assume that your expressions are defined for all x in the domain of the composition. It is a good idea to write the domain of each of the functions to make sure you know what x values make sense in the functions.	
a. $f(x) = 6x + 3$ and $g(x) = 6x - 3$ $f(g(x))$ $g(f(x))$ ○ f and g are inverses of each other ○ f and g are not inverses of each other	b. $f(x) = \frac{2}{x}$ and $g(x) = \frac{2}{x}$ $f(g(x))$ $g(f(x))$ ○ f and g are inverses of each other ○ f and g are not inverses of each other
c. $f(x) = \sqrt{x}$ and $g(x) = x^2$, for $x \geq 0$ $f(g(x))$ $g(f(x))$ ○ f and g are inverses of each other ○ f and g are not inverses of each other	d. $f(x) = \sqrt{x}$ and $g(x) = x^2$ $f(g(x))$ $g(f(x))$ ○ f and g are inverses of each other ○ f and g are not inverses of each other

5. Evaluate the following given the one-to-one functions below.			
	e. $f(1)$ f. $f^{-1}(1)$ g. $f(2)$ h. $f^{-1}(-4)$ i. $f(f^{-1}(-3))$ j. $f^{-1}(f(-4))$		e. $f(2)$ f. $f^{-1}(0)$ g. $f(-5)$ h. $f^{-1}(-2)$ i. $f(f^{-1}(-2))$ j. $f^{-1}(f(-5))$

6. Find the inverses of the following one-to-one functions. Then find the domains and ranges of the functions and their inverses.

$f(x) = \dfrac{7x+1}{2x-1}$		$g(x) = \sqrt{2x-1}$ for $x \geq \frac{1}{2}$	
Domain of f	Range of f^{-1}	Domain of g	Range of g^{-1}
Domain of f^{-1}	Range of f	Domain of g^{-1}	Range of g

$h(x) = 2^x$		$p(x) = lnx$	
Domain of h	Range of h^{-1}	Domain of p	Range of p^{-1}
Domain of h^{-1}	Range of h	Domain of p^{-1}	Range of p