1.3 Exponential and Logarithmic Functions

We introduced the basic concept of the exponential and logarithmic functions as inverses of each other.

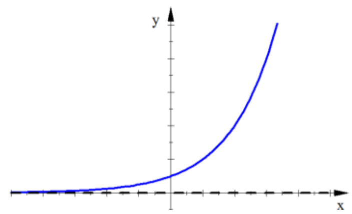
Review

1. **Exponential Function:** An exponential function is defined as $f(x) = a^x$, where a is a positive real number not equal to 1.

Properties of an exponential function
- Domain: All real numbers
- Range: $(0, \infty)$
- One-to-One: Yes!
- Points on the graph to plot are

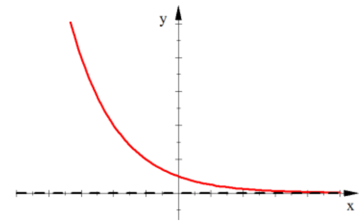
x	$y = a^x$
-2	$a^{-2} = \frac{1}{a^2}$
-1	$a^{-1} = \frac{1}{a}$
0	$a^0 = 1$
1	$a^1 = a$
2	a^2

Graph of $f(x) = a^x$, for $a > 1$



Horizontal Asymptote $y = 0$

Graph of $f(x) = a^x$, for $a < 1$

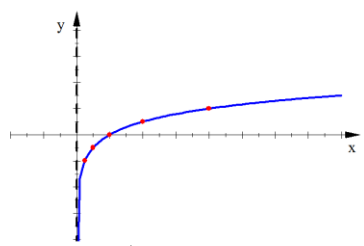


Horizontal Asymptote $y = 0$

2. **Logarithmic Function:** A function defined as $f(x) = log_a x$, where $a > 0,\ a \neq 1$ and is positive real number.

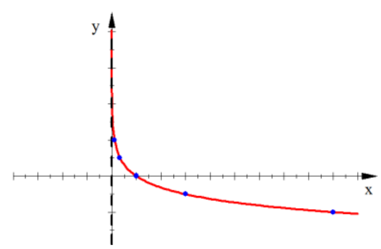
Properties of the logarithmic function:
- To compute the value of $f(x)$ means answering: $a^? = x$
- Graph:
- Domain:$(0, \infty)$
- Range: All real numbers
- One-to-One: Yes!
- Inverse function $f^{-1}(x) = a^x$

Graph of $f(x) = log_a x$, for $a > 1$



Vertical Asymptote $x = 0$

Graph of $f(x) = log_a x$, for $0 < a < 1$



Vertical Asymptote $x = 0$

Let's work more with these functions and observe their properties so we can use them in many different applications.

Remember the fact that the logarithmic function is the inverse function of the exponential function. This fact means the equations below are equivalent.

Exponential Equation	$\Leftrightarrow$	**Logarithmic Equation**
$\boldsymbol{y = a^x}$	$\Leftrightarrow$	$\boldsymbol{log_a y = x}$

There some special numbers we will be working with as bases for logarithmic and exponential functions and they get special notations. For example when the base is 10 we write $log_{10}x = logx$ the base is left out. Check out your calculators, you should see a log button on it. This represents the logarithm base ten.

So $y = 10^x \iff logy = x$

The notation of logarithms originated perhaps sometime in the mid 1600's. The motivation for working with logarithms base 10 comes from the ease with which they allows us to work with very large numbers (astronomical scale) or very small numbers (microscopic scale).

Below are examples where in real life logarithms base ten are used.

1. In chemistry pH of a substance is measured by the formula $pH = -log[H^+]$ where $[H^+] =$ hydrogen ions concentration in units of moles/liter. A substance having pH of less than 7 is characterized as being acidic and above 7 to be basic. Water has a pH of 7. That means $7 = -log[H^+]$ or that 1×10^{-7} moles/liter of hydrogen ions are present in water. The pH of a orange juice is between 3 and 4, while sweet corn has a pH of about 8.

   Nutrition experts talk about the link of alkaline diets [it is the alkaline ash producing foods (ash is the residue left by the food after combustion), so even some citrus foods are alkaline ask producing foods] reducing risk of cancer which can be accomplished with eating a lot of fruits and vegetables.

2. Seismologist's measure earthquakes using the Richter scale. The formula for this is given by $M = log\left(\frac{I}{I_0}\right)$ where $I =$ Intensity or amplitude of the earthquake as measured by the seismograph usually located within 100 kilometers from the epicenter, and $I_0 =$ Intensity or amplitude of the earth's vibrations when there is no earthquake. To get a sense of this scale a Richter scale reading of a 3 is similar to the vibrations felt when a large truck is driving by. The energy released by a magnitude 5 earthquake is only equivalent to 500 tons of TNT explosion where as a magnitude 9 earthquake that hit Japan in 2011 was about 500,000,000 tons of TNT. So it gives you a sense of what happens when we are working with powers of ten. That 2011 earthquake that hit Japan was so powerful it knocked our earth off its axis so that the days on earth shortened by about 1.8 microseconds.
3. Physicists use a similar formula to calculate the decibel level of sound $dB = 10\ log\left(\frac{I}{I_0}\right)$ again $I =$ Intensity or amplitude of the sound heard measured in $Watts/meter^2$, and $I_0 =$ lowest threshold for sound that can be heard and is about $10^{-12}\ Watts/meter^2$ (like a whisper). So 60 decibels is like when you are in a restaurant and you hear the noises around you, between 130-150 decibels are is the sound of a jet taking off (this kind of high noise exposure can lead to hearing loss).

Another number we will be working with is "e" or is sometimes to refered to as Euler's number or Napier's constant and therefore the letter e is used to denote it. This special number came about in the mid 1600s' and was first noticed by Bernoulli when working with compound interest. You did some problems as shown below in the previous section. Let's play with these a little and see what happens.

Compound interest problem-

Suppose that you invested a $1000\$$ at an interest of 5% compounded say n times a year. That means that if interest is compounded a) Yearly $n = 1$, b) Quarterly $n = 4$, c) Monthly $n = 12$, d) Weekly $n = 52$, and so on …

Remember our principle remains the same so when our interest is compounded yearly

After 1 year we will have our principle + interest

$= 1000 + 1000(0.05) = 1000(1 + 0.05) = 1000(1.05) = 1050\$$

After 2 years we will have our principle of 1050 plus interest $= 1050 + 1050(0.05)$

$$= 1050(1 + 0.05) = 1050(1.05) = 1000(1.05)(1.05) = 1000(1.05)^2 = 1102.50\$$$

We see then by extending our ideal that amount of money after t years will be $1000(1.05)^t$ dollars. What happens though if the interest is paid quarterly in the same manner? Since interest is paid 4 times a year and the rate of 5% is divided equally in 4 we get

After the first quarter we will make $1000 + 1000\left(\frac{0.05}{4}\right) = 1000\left(1 + \frac{0.05}{4}\right)^1 = 1012.50$ after the second quarter we will make $1000\left(1 + \frac{0.05}{4}\right)^2 = 1025.16$ and so after one year we will make $1000\left(1 + \frac{0.05}{4}\right)^4 = 1050.95$ so you can see we make slightly more money when interest is paid quarterly. Now after two years we will have 8 pay periods so we would have $1000\left(1 + \frac{0.05}{4}\right)^{4\times 2} = 1104.49\$$.

A natural question to ask is, what if we keep on increasing the number of times interest is paid to us in a year, would we keep on making more and more money?

In general then we will have $1000\left(1 + \frac{0.05}{n}\right)^{nt}$ where n =number of pay periods per year, and t =number of years the money was invested for.

Using this formula the chart below will follow what happens to 1000 dollars when we change the number of pay periods, and the number of years.

n pay periods in a year		t years later				
		1	2	3	10	15
Yearly	1	1050	1102.50	1157.63	1628.89	2078.93
Semi annually	2	1050.63	1103.81	1159.69	1638.62	2097.57
Quarterly	4	1050.95	1104.49	1160.75	1643.62	2107.18
Monthly	12	1051.16	1104.94	1161.47	1647.01	2113.70
Weekly	52	1051.25	1105.12	1161.75	1648.32	2116.24
Per day	365	1051.26	1105.16	1161.82	1648.66	2116.89
Per hour (365×24)	8760	1051.26	1105.16	1161.82	1648.66	2116.89
Per minute $(365 \times 24 \times 60)$	525600	1051.27	1105.17	1161.83	1648.72	2117.00
Per second $(365 \times 24 \times 60 \times 60)$	31536000	1051.27	1105.17	1161.83	1648.72	2117.00

As a mathematician this table should make you ponder a bit as we would have expected our values to keep getting higher and higher when the number of pay periods increase. We wonder what happens as n goes to infinity or $n \to \infty$. We see that eventually it does not even make a dent in the money we make. This would suggest that the numbers probably will start to accumulate closer and closer to the same number no matter how many times interest is paid in a year. This is when we have to put our thinking hat on and explore the number of the type $\left(1 + \frac{1}{n}\right)^n$ as $n \to \infty$. Make a table like above and explore this yourself before we give you the answer, use your calculator or go to www.wolframalpha.com and you will be able to see many more digits on display than a normal calculator. Once you have done so without looking it up on your own make a prediction of what you think this number is below $\left(1+\frac{1}{n}\right)^n \xrightarrow{n\to\infty}$ ___________________ (do this before going to next page)

What you should have seen is that the number $(1+\frac{1}{n})^n$, where n goes to infinity is that it approaches an irrational number mathematicians called e. The first digits of e are 2.718…. See some of the digits for particular n values below.

n	$(1+\frac{1}{n})^n$
100	2.704813829
1000	2.716923932
10000	2.718145927
100000	2.718268237
1000000	2.718280469
10000000	2.718281692

This number is used in many applications and in fact in mathematics this number is called a transcendental number.

A **transcendental number** is a real or complex number that is not algebraic - or one that is not a root to a polynomial equation with rational coefficients.

You already know another transcendental number π. The complex number i however is an algebraic number, since i is a solution to the equation $x^2+1=0$. The number $\sqrt{2}$ is also an algebraic number. Can you figure out why?

So if we let the number of pay periods in a compound interest problem go to infinity we can write that in the following formula $1000\left(1+\frac{0.05}{n}\right)^{nt} \xrightarrow{n\to\infty} 1000e^{0.05t}$, the continuous compound interest formula.

In general, the formula $A = Pe^{\pm rt}$ is then also used to describe most exponential growth and decay where r is the growth rate or decay rate. The sign for the exponent is negative in case of decay. We will work in these applications in more detail later.

A natural logarithm represented as $lnx = log_e x$ is the inverse function of the exponential function, $y = e^x$.

Review of equivalent logarithmic and exponential equations

Exponential Equation	$\Leftrightarrow$	**Logarithmic Equation**
$\boldsymbol{y = a^x}$	$\Leftrightarrow$	$\boldsymbol{log_a y = x}$
$\boldsymbol{y = 10^x}$	$\Leftrightarrow$	$\boldsymbol{logy = x}$
$\boldsymbol{y = e^x}$	$\Leftrightarrow$	$\boldsymbol{lny = x}$

Practice Problems

1. Convert the exponential equations below into their equivalent logarithmic form.

Exponential Equation	Equivalent Logarithmic Equation
a. $8 = 2^3$	
b. $\frac{1}{8} = 2^{-3}$	
c. $10 = 2^x$	
d. $5^x = 25$	
e. $\left(\frac{1}{5}\right)^x = 125$	
f. $3^{2x-4} = 81$	
g. $0.4^x = 5$	
h. $e^x = 6$	
i. $10^x = 0.0001$	
j. $e^{x+1} = 0.3$	

2. Convert the logarithmic equations below into their equivalent exponential form.

Logarithmic Equation	Equivalent Exponential Equation
a. $log_3 x = -2$	
b. $lnx = 5$	
c. $\log(x+1) = 5$	
d. $-3 = logx$	
e. $1.5 = lnx$	
f. $5 = log_2 x$	
g. $4 = log_{\frac{1}{2}}(x)$	
h. $log_5 25 = 2$	
i. $log_3\left(\frac{1}{81}\right) = -4$	

Practice Problems Solutions

It is helpful when doing these problems to identify the base in the problem. Remember the side opposite the one that has the base in an exponential equation inherits the logarithmic function, and the side opposite the one that has the logarithm inherits the base raised to the appropriate exponent.

In general then, $\boldsymbol{Base}^{\boldsymbol{Exponent}} = quantity \Leftrightarrow \boldsymbol{Exponent} = log_{\boldsymbol{Base}}(quantity)$ and

1. Convert the exponential equations below into their equivalent logarithmic form.

Exponential Equation	Equivalent Logarithmic Equation
a. $8 = 2^3$ Base is $\mathbf{2}$ Exponent is $\mathbf{3}$	a. $log_2 8 = \mathbf{3}$
b. $\frac{1}{8} = 2^{-3}$ Base is $\mathbf{2}$ Exponent is $\mathbf{-3}$	b. $log_2\left(\frac{1}{8}\right) = -3$
c. $10 = 2^x$ Base is $\mathbf{2}$ Exponent is $\mathbf{10}$	c. $log_2 10 = x$
d. $5^x = 25$ Base is $\mathbf{5}$ Exponent is $\boldsymbol{x}$	d. $x = log_5(25)$
e. $\left(\frac{1}{5}\right)^x = 125$ Base is $\mathbf{\frac{1}{5}}$ Exponent is $\boldsymbol{x}$	e. $x = log_{\frac{1}{5}}(125)$
f. $3^{2x-4} = 81$ Base is $\mathbf{3}$ Exponent $\boldsymbol{2x-4}$	f. $2x - 4 = log_3 81$
g. $0.4^x = 5$ Base is $\mathbf{0.4}$ Exponent is $\boldsymbol{x}$	g. $x = log_{0.4}(5)$
h. $e^x = 6$ Base is $\boldsymbol{e}$ Exponent is $\boldsymbol{x}$	h. $x = ln6$
i. $10^x = 0.0001$ Base is $\mathbf{10}$ Exponent is $\boldsymbol{x}$	i. $x = \log(0.0001)$
j. $e^{x+1} = 0.3$ Base is $\boldsymbol{e}$ Exponent is $\boldsymbol{x+1}$	j. $x + 1 = \ln(0.3)$

2. Convert the logarithmic equations below into their equivalent exponential form.

In general here we will use $\boldsymbol{Exponent} = log_{\boldsymbol{Base}}(quantity) \Leftrightarrow \boldsymbol{Base}^{\boldsymbol{Exponent}} = quantity$

Logarithmic Equation	Equivalent Exponential Equation
a. $log_3 x = -2$ Base is $\mathbf{3}$ Exponent is $\mathbf{-2}$	a. $x = \mathbf{3}^{-2} = \frac{1}{9}$
b. $lnx = 5$ Base is $\boldsymbol{e}$ Exponent is $\mathbf{5}$	b. $x = \boldsymbol{e}^{\mathbf{5}}$
c. $\log(x+1) = 5$ Base is $\mathbf{10}$ Exponent is $\mathbf{5}$	c. $x + 1 = \mathbf{10}^{\mathbf{5}}$ or $x =$ 9999
d. $-3 = logx$ Base is $\mathbf{10}$ Exponent is $\mathbf{-3}$	d. $\mathbf{10}^{-3} = x$ or $x = 0.001$
e. $1.5 = lnx$ Base is $\boldsymbol{e}$ Exponent is $\mathbf{1.5}$	e. $\boldsymbol{e}^{\mathbf{1.5}} = x$
f. $5 = log_2 x$ Base is $\mathbf{2}$ Exponent is $\mathbf{5}$	f. $\mathbf{2}^{\mathbf{5}} = x$ or $32 = x$
g. $4 = log_{\frac{1}{2}}(x)$ Base is $\frac{\mathbf{1}}{\mathbf{2}}$ Exponent is $\mathbf{4}$	g. $\left(\frac{\mathbf{1}}{\mathbf{2}}\right)^{\mathbf{4}} = x$ or $\frac{1}{16} = x$
h. $log_5 25 = 2$ Base is $\mathbf{5}$ Exponent is $\mathbf{2}$	h. $25 = \mathbf{5}^{\mathbf{2}}$
i. $log_3\left(\frac{1}{81}\right) = -4$ Base is $\mathbf{3}$ Exponent is $\mathbf{-4}$	i. $\frac{1}{81} = \mathbf{3}^{\mathbf{-4}}$

Given the fact that any real number raised to an exponent always produces a positive number, the domain of a logarithmic function is always all real values so that the input is a positive number.

Practice Problems

1. Find the domain of the functions below.
   a. $f(x) = log_2(x-1)$
   b. $g(x) = \ln(2-x)$
   c. $h(x) = \log(1+2x)$

Attempt these first before looking on the next page for solutions.

Solutions to Practice Problems

1. Find the domain of the functions below.
    a. $f(x) = log_2(x-1)$
    Since the input has to be non-zero we must have $x-1>0$ or $x>1$
    Domain is $(1,\infty)$

    b. $g(x) = \ln(2-x)$
    Since the input has to be non-zero we must have $2-x>0$ or $2>x$
    Domain is $(-\infty,2)$

    c. $h(x) = \log(1+2x)$
    Since the input has to be non-zero we must have $1+2x>0$ or $1> -2x$ or $-\frac{1}{2}<x$
    Domain is $\left(-\frac{1}{2},\infty\right)$

Evaluating Logarithmic Functions

Evaluating $log_a x$ means asking the question $a^? = x$.

Practice Problems

Evaluate the following

1. $log_2(16)$
2. $log_{\frac{1}{5}}(25)$
3. $log1000$
4. lne^3
5. $log_9 3$

Attempt these problems yourself and then look at the answers.

Solutions to Practice Problems

It would help a lot if again you can identify the base, then the exponent is what you are looking for.

Evaluate the following

1. $log_2(16)$

   Base $= 2$ and so we are looking for $2^? = 16$ and we know that $16 = 2^4$ so we have $2^? = 2^4$
   Since exponential functions are one-to-one the only way this can happen is if we have $? = 4$.
   Therefore $log_2(16) = 4$

2. $log_{\frac{1}{5}}(25)$
3. Base $= \frac{1}{5}$ and so we are looking for $\left(\frac{1}{5}\right)^? = 25$ and we know that $25 = 5^2 = \left(\frac{1}{5}\right)^{-2}$ so we have $\left(\frac{1}{5}\right)^? = \left(\frac{1}{5}\right)^{-2}$ Since exponential functions are one-to-one the only way this can happen is if we have $? = -2$. Therefore, $log_{\frac{1}{5}}(25) = -2$.
4. $log1000 = 3$ since $10^3 = 1000$
5. $lne^3 = 3$ since $e^3 = e^3$
6. $log_9 3 = \frac{1}{2}$ since $9^{\frac{1}{2}} = 3$

Because logarithms deal with exponents we have properties of logarithmic functions that we will now investigate.

Properties of Logarithms

Playing

Now that we are more comfortable with logarithms and see them as exponents we may wonder how these affect properties of logarithms. Let us play and see what happens.

Let $x = log_a u,\ y = log_a v$ and $a, b > 0, a, b \neq 1$, a real number and n is any real number. We know from changing logarithmic equations to exponential equations that we have $a^x = u$, and $a^y = v$.

1. $a^x a^y = a^{x+y}$ by using properties of exponents since the base is the same. Therefore we have $uv = a^{x+y}$. This means that if we change the exponential equation to logarithmic equation we get $log_a(uv) = x + y$ or that

   $$log_a(uv) = log_a(u) + log_a(u)$$

2. $\frac{a^x}{a^y} = a^{x-y}$ from properties of exponents since the base is the same. Therefore we have $\frac{u}{v} = a^{x-y}$ that means if we change the exponential equation to logarithmic equations we get $log_a\left(\frac{u}{v}\right) = x - y$ or that

   $$log_a\left(\frac{u}{v}\right) = log_a(u) - log_a(u)$$

3. $a^0 = 1$ that means

   $$log_a(1) = 0$$

4. $(a^x)^n = a^{nx}$ from properties of exponents. Therefore we have $(u)^n = a^{nx}$ or that $log_a((u)^n) = nx$ and since $x = log_a u$

   $$log_a(u^n) = nlog_a u$$

5. What about $\frac{log_a u}{log_a b}$ what is that equal to?

   Since we do not know the value of $\frac{log_a u}{log_a b}$, we can starting with writing $z = \frac{log_a u}{log_a b}$.

   This gives us - $zlog_a b = log_a u$. Then by applying the previous property we have $log_a b^z = log_a u$. Since logarithm is a one-to-one function the two sides are same if and only if $b^z = u$. So that gives that $z = log_b u$. That means $\frac{log_a u}{log_a b} = log_b u$

   This may not seem like an interesting fact but what this means is that we can now evaluate the logarithms even when the base is something other than a ten or an "e".

   For example, we can now calculate $log_3 5$ approximately anyway using our calculators since $log_3 5 = \frac{log5}{log3} = \frac{ln5}{ln3} = \frac{log_{34}5}{log_{34}3}$. The individual numerator and denominator will be different values but the ratio is the same. Check these numbers for yourself.

   $log5 = 0.69897$ but $ln5 = 1.60943791$, and $log3 = 0.47712125$ but $ln3 = 1.09861228$. But look $\frac{log5}{log3} = 1.464973521$ and $\frac{ln5}{ln3} = 1.464973521$.

   This formula is referred to as the change of base formula and we write it as

   $$\frac{log_a u}{log_a b} = log_b u$$

We can use these properties to simplify our logarithmic terms.

Practice Problems

1. Properties of logarithms: Please fill out the missing values.	
A. $log_2 7 + log_2 9 =$ ____________________	B. $log_a r - log_a s =$ ____________________
C. $log_a(st) =$ ____________________	D. $log_a\left(\frac{s}{t}\right) =$ ________________
E. $log_a(r^n) =$ ____________________	F. D $nlog_a(r) =$ ____________________
G. Change of base formula: $log_a(r) = \frac{}{}$	

2. Use properties of logarithms to do the problems below.

A. Fill in the missing values to make the statement a true statement.

i. $log_7 12 - log_7(\_\_\_\_\_) = log_7 4$

ii. $log_6 5 + log_6 8 = log_6(\_\_\_\_\_)$

iii. $-2log_5 3 = log_5(\_\_\_\_)$

iv. $log_3 16 = (\_\_\_\_\_)log_3 2$

v. $\frac{ln5}{ln6} = log_6(\_\_\_\_\_)$

B. Expand the following. Each logarithm in your answer should involve only one variable. Assume that all variables are positive.

I. $\log(x^5y^7) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

II. $\log_3\left(\frac{x^7y^3}{\sqrt[3]{z}}\right) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

III. $\log\left(\frac{x^5}{\sqrt{z^7y^3}}\right) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

IV. $\ln((4-x)(x+2)) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

V. $\ln\left(\frac{x^6\sqrt{7}}{5z^2}\right) = \_\_\_\_\_\_\_\_\_\_\_\_$

C. Write the following as one term.

i. $6log_3 x + 3log_3 y = \_\_\_\_\_\_\_\_\_\_$

ii. $\frac{1}{2}logx - 3logy - 2logz = \_\_\_\_\_\_\_\_\_\_$

D. Compute the values below exactly

i. $log_2\left(\frac{1}{8}\right) = \_\_\_\_\_\_\_\_\_\_$

ii. $\log(0.000001) = \_\_\_\_\_\_\_\_\_\_$

iii. $\ln(e^6) = \_\_\_\_\_\_\_\_\_\_$

iv. $-\ln(\sqrt{e}) = \_\_\_\_\_\_\_$

v. $log_3\left(\frac{1}{27}\right) = \_\_\_\_\_\_\_\_$

E. Use a scientific calculator evaluate the following, and round your answers to 2 digits.

i. $\frac{2000}{\log(1+\frac{0.04}{5})} = \_\_\_\_\_\_\_\_\_\_$

ii. $\frac{log5}{log3} = \_\_\_\_\_\_\_\_$

iii. $\frac{\ln(0.03)}{2} = \_\_\_\_\_\_\_\_$

iv. $log_5 4 = \_\_\_\_\_\_\_\_$

Practice Problems Solutions

1. Properties of logarithms: Please fill out the missing values.

A. $log_2 7 + log_2 9 = log_2(7 \times 9) = log_2 63$

B. $log_a r - log_a s = log_a\left(\frac{r}{s}\right)$

C. $log_a(st) = log_a s + log_a t$

D. $log_a\left(\frac{s}{t}\right) = log_a s - log_a t$

E. $log_a(r^n) = nlog_a(r)$

F. $nlog_a(r) = log_a(r^n)$

G. Change of base formula: $log_a(r) = \frac{logr}{loga} = \frac{lnr}{lna}$

2. Use properties of logarithms to do the problems below.

A. Fill in the missing values to make the statement a true statement.

i. $log_7 12 - log_7(3) = log_7 4$

ii. $log_6 5 + log_6 8 = log_6(40)$

iii. $-2log_5 3 = log_5\left(\frac{1}{3^2}\right) = log_5\left(\frac{1}{9}\right)$

iv. $log_3 16 = (4)log_3 2$

v. $\frac{ln5}{ln6} = log_6(5)$

B. Expand the following. Each logarithm in your answer should involve only one variable. Assume that all variables are positive.

I. $\log(x^5y^7) = 5logx + 7logy$

II. $\log_3\left(\frac{x^7y^3}{\sqrt[3]{z}}\right) = 7\log_3 x + 3\log_3 y - \frac{1}{3}\log_3 z$

III. $\log\left(\frac{x^5}{\sqrt{z^7y^3}}\right) = 5\log_3 x - \frac{1}{2}(7\log_3 z + 3\log_3 7)$

$= 5\log_3 x - \frac{7}{2}\log_3 z - \frac{3}{2}\log_3 y$

IV. $\ln((4-x)(x+2)) = ln(4-x) + ln(x+2)$

V. $\ln\left(\frac{x^6\sqrt{7}}{5z^2}\right) = 6\ln x + \frac{1}{2}ln7 - (ln5 + 2lnz)$

$= 6\ln x + \frac{1}{2}ln7 - ln5 - 2lnz$

C. Write the following as one term.

i. $6log_3 x + 3log_3 y = log_3(x^6y^3)$

ii. $\frac{1}{2}logx - 3logy - 2logz = log\left(\frac{\sqrt{x}}{y^3z^2}\right)$

D. Compute the values below exactly

i. $log_2\left(\frac{1}{8}\right) = -3$

ii. $\log(0.000001) = -6$

iii. $\ln(e^6) = 6$

iv. $-\ln(\sqrt{e}) = -\frac{1}{2}$

v. $log_3\left(\frac{1}{27}\right) = -3$

E. Use a scientific calculator evaluate the following, and round your answers to 2 digits.

i. $\frac{2000}{\log\left(1+\frac{0.04}{5}\right)} = 250998.67$

ii. $\frac{log5}{log3} = 1.46$

iii. $\frac{\ln(0.03)}{2} = -1.75$

iv. $log_5 4 = \frac{log4}{log5} = 0.86$

Cover Sheet for Sec 1.3a   Name: ______________________________

Summary

Concept	In your words describe what these concepts mean to you
Review Properties of logarithms	
What is the use of the logarithms and exponential functions?	
How are exponential and logarithmic functions related to each other?	
How do we find domains of logarithmic functions?	
In your own words describe how to change logarithmic equations into an exponential equation.	
In your own words describe how to change exponential equations into logarithmic equations.	

Video Log 1.3a

1. Rewrite the exponential equations in logarithmic form and logarithmic equations in exponential form. If possible simplify your answers.

Exponential Equation	Logarithmic Equation
$e^x = 5$	
$2^{x+1} = 8$	
	$log_2(x) = -1$
	$\log(x+1) = 2$
	$\ln(x+1) = 3$
$5^{1-x} = 3$	
	$log_{\frac{1}{2}}(x) = -3$

Exponential Equation	Logarithmic Equation
	$log_{\frac{1}{3}}(81) = -4$
$5^2 = 25$	
	$log_3\left(\frac{1}{9}\right) = -2$
$e^3 = x$	
	$\ln(e^2) = 2$
$10^{-2} = 0.01$	
	$log1000 = 3$

2. Find the domain of the functions below.

A. $f(x) = \log(x+1)$ Domain:	B. $g(x) = log\left(\frac{3}{x-4}\right)$ Domain:	C. $h(x) = 3^{x-1}$ Domain:	D. $h(x) = ln(1-x)$ Domain:

3. Properties of logarithms: Please fill out the missing values.

A.

$log_a x + log_a y =$ ____________________

B.

$log_a x - log_a y =$ ____________________

C.

$log_a(xy) =$ ____________________

D.

$log_a\left(\frac{x}{y}\right) =$ ________________

E.

$log_a(x^n) =$ ____________________

F. D

$nlog_a(x) =$ ____________________

G. Change of base formula:

$$log_a(x) = \_\_\_\_\_\_\_\_\_\_$$

4. Use properties of logarithms to do the problems below.

F. Fill in the missing values to make the statement a true statement.

vi. $log_5 8 - log_5(\_\_\_\_\_\_) = log_5 4$

vii. $log_2 3 + log_2 5 = log_2(\_\_\_\_\_\_)$

viii. $3log_7 2 = log_7(\_\_\_\_)$

ix. $log_5 49 = (\_\_\_\_\_\_)log_5 7$

x. $\frac{ln5}{ln4} = log_4(\_\_\_\_\_\_)$

G. Expand the following. Each logarithm in your answer should involve only one variable. Assume that all variables are positive.

VI. $\log(x^3y^2) =$______________________

VII. $\log_2\left(\frac{x^3y^2}{\sqrt{z}}\right) =$______________________

VIII. $\log\left(\frac{x^3}{\sqrt{z^5y}}\right) =$______________________

IX. $\ln((4+x)(x-2)) =$______________________

X. $\ln\left(\frac{x^5\sqrt[3]{y}}{3z}\right) =$ ________________

H. Write the following as one term.

i. $4log_2 x + 2log_2 y =$ ____________

ii. $\frac{1}{3}logx - 2logy + 3logz =$ ____________

J. Use a scientific calculator to evaluate the following, and round your answers to 2 digits.

i. $\frac{20000}{\log(1+\frac{0.02}{5})} =$ ____________

ii. $\frac{log3}{log2} =$ __________

iii. $\frac{\ln(0.5)}{3} =$ __________

iv. $log_3 5 =$ __________

I. Compute the values below exactly

i. $log_2(8) =$ ____________

ii. $\log(0.000001) =$ ____________

iii. $\ln(e^5) =$ ___________

iv. $\ln(\sqrt{e}) =$ ________

v. $log_5\left(\frac{1}{25}\right) =$ __________

More Types of Functions

Sequences

A sequence is an ordered collection of objects. Repetitions are allowed.

If you take all the real valued functions and let the domain be restricted to a set of integers, then taking the collection of the real valued outputs of these functions is one way to generate interesting sequences.

Examples

1. Let the constant function $f(n) = 50$ for all natural numbers n. Let us label $f(n) = a_n$. Then $a_n = 50$, for $n = 1,2,3,4,\ldots$ is a sequence of numbers with all outputs being 50. This is called a constant sequence. So when we write a_{43}, it is referring to the term in the sequence that is the 43rd number in the sequence. In our case it would be 50 since all output is 50.
2. Let our terms be defined by a linear function restricted to the domain of whole numbers say $a_n = 3 + 7t$ that would mean every subsequent term is 7 higher than the previous term. Or a 7 gets added to each consecutive term starting with the first term which would be a 3.
   $a_0 = 3, a_1 = 10, a_2 = 17, a_3 = 24, a_4 = 31,\ldots.$
   If asked to find the 50th term since we started at zero we would have $a_{49} = 3 + 49(7) = 346$
   Since each subsequent term is found by adding a 7 the previous term, such a sequence is called an arithmetic sequence. You can also think of how the difference of two consecutive terms is a constant.
3. Let our terms be defined by an exponential function restricted to the domain of whole numbers say $a_n = 3(2)^n$ that would mean every subsequent term is a multiple of 2 from the previous term. Or a 2 gets multiplied to each consecutive term starting with the first term which would be a 3.
   $a_0 = 3, a_1 = 6, a_2 = 12, a_3 = 24, a_4 = 48,\ldots.$
   If asked to find the 50th term since we started at zero we would have $a_{49} = 3(2)^{49}$ (this is too large of a number to multiply out)
   Since each subsequent term is found by multiplying by a 2, such a sequence is called a geometric sequence. You can also think of this as ratio of two consecutive terms is the same.

Practice Problems

1. For the sequences below determine if they are arithmetic or geometric. Then find the formula for the a_n and then fill in the missing terms column with value of that term.

Sequence	Type	$f(n) = a_n = n^{th}$ term
$\{13,17,21,25,\ldots.\}$	☐ Arithmetic ☐ Geometric ☐ Neither	
$\{7,21,63,189,\ldots\}$	☐ Arithmetic ☐ Geometric	

	☐ Neither	
$\{12,10,7,1, \ldots.\}$	☐ Arithmetic ☐ Geometric ☐ Neither	
$\{1,-11,1,-11, \ldots.\}$	☐ Arithmetic ☐ Geometric ☐ Neither	

2. Find the first 4 terms of the sequences given below.

$a_n = n^{th}$ term of the sequence	First term	Second term	Third term	Fourth term
$a_n = 5\left(\frac{1}{2}\right)^{2n+1}, n = 0,1,2, \ldots$				
$a_n = 4n + 3, n = 4,5, \ldots$				
$a_n = \frac{2n-1}{n+2}, n = 1,2,3, \ldots$				
$a_n = \frac{(-1)^n}{2n}, n = 1,2, \ldots$				

3. For a given arithmetic sequence, the 82^{nd} term, $a_{82} = -373$ and the 6^{th} term, $a_6 = 7$. Find the 42nd term a_{42}.

4. For a given geometric sequence, the 7^{th} term, $a_7 = 15$ and the 9^{th} term, $a_9 = 135$. Find the 10^{th} term a_{10}.

Practice Problems Solutions

1. For the sequences below determine if they are arithmetic or geometric. Then find the formula for the a_n and then fill in the missing terms column with value of that term.

Sequence	Type	$f(n) = a_n = n^{th}$ term
$\{13,17,21,25, \ldots.\}$ Since the difference between each term is 4	✓ Arithmetic Since the difference between each term is 4	$a_n = 13 + 4n, n = 0,1,2,3, \ldots$
$\{7,21,63,189, \ldots\}$	✓ Geometric Since the ratio of two consecutive terms is 3.	$a_n = 7(3)^n, n = 0,1,2,3, \ldots$
$\{12,10,7,1, \ldots.\}$	✓ Neither	

	The difference or ratio of two consecutive terms is not the same.	
$\{1,-11,1,-11,\ldots.\}$	✓ Neither Terms just alternate between 1 and -11.	

2. Find the first 4 terms of the sequences given below.

$a_n = n^{th}$ term of the sequence	First term	Second term	Third term	Fourth term
$a_n = 5\left(\frac{1}{2}\right)^{2n+1}, n = 0,1,2,\ldots$	$a_0 = \frac{5}{2}$	$a_1 = \frac{5}{8}$	$a_2 = \frac{5}{32}$	$a_3 = \frac{5}{128}$
$a_n = 4n + 3, n = 4,5,\ldots$	$a_4 = 19$	$a_5 = 23$	$a_6 = 27$	$a_7 = 31$
$a_n = \frac{2n-1}{n+2}, n = 1,2,3,\ldots$	$a_1 = \frac{1}{3}$	$a_2 = \frac{3}{4}$	$a_3 = \frac{5}{5} = 1$	$a_4 = \frac{7}{6}$
$a_n = \frac{(-1)^n}{2n}, n = 1,2,\ldots$	$a_1 = -\frac{1}{2}$	$a_2 = \frac{1}{4}$	$a_3 = -\frac{1}{6}$	$a_4 = \frac{1}{8}$

3. For a given arithmetic sequence, the 82^{nd} term, $a_{82} = -373$ and the 6^{th} term, $a_6 = 7$. Find the 42^{nd} term a_{42}.

We know that if it is arithmetic sequence to get from 6^{th} term to 82^{nd} term we would have to do the following

$a_{82} = -373 = 7 + d(82 - 6)$ where $d =$ common difference between two consecutive terms. Solving for d we get $-373 - 7 = d(76)$

$$d = \frac{-380}{76} = -5$$

To get the 42^{nd} term we will have to add

$$a_{42} = 7 + (-5)(42 - 6) = -173$$

4. For a given geometric sequence, the 7^{th} term, $a_7 = 15$ and the 9^{th} term, $a_9 = 135$. Find the 10^{th} term a_{10}.

We know geometric sequence must have same ratio of two consecutive terms

So $a_9 = 135 = a_7(r)^2 = 15(r)^2$

Solving for r we get

$(r)^2 = \frac{135}{15} = 9$ so $r = 3$ since the terms are all positive.

$$a_{10} = 3(135) = 405$$

Cover Sheet for Sec 1.3b **Name: ______________________________**

Summary

Concept	In your words describe what these concepts mean to you
What is a sequence?	
What is an arithmetic sequence?	
What is a geometric sequence?	
Give an example of a sequence that is neither an arithmetic or a geometric sequence.	
What is a Fibonacci sequence?	
What is the golden ratio?	

Video Log 1.3b

1. For the sequences below determine if they are arithmetic or geometric. Then find the formula for the a_n and then fill in the missing terms column with value of that term.

Sequence	Type	$f(n) = a_n = n^{th}$ term
$\{22,26,30,34, \ldots.\}$	☐ Arithmetic ☐ Geometric ☐ Neither	
$\{6,12,24, \ldots\}$	☐ Arithmetic ☐ Geometric ☐ Neither	
$\{9,13,17, \ldots.\}$	☐ Arithmetic ☐ Geometric ☐ Neither	
$\{7,11,19, \ldots.\}$	☐ Arithmetic ☐ Geometric ☐ Neither	

2. Find the first 4 terms of the sequences given below.

$a_n = n^{th}$ term of the sequence	First term	Second term	Third term	Fourth term
$a_n = 4\left(\frac{1}{3}\right)^{2n+1}, n = 0,1,2, \ldots$				
$a_n = 3n + 5, n = 3,4,5, \ldots$				
$a_n = \frac{2n}{n+3}, n = 3,4,5, \ldots$				
$a_n = \frac{(-1)^n}{n}, n = 1,2, \ldots$				

3. For a given arithmetic sequence, the 82^{nd} term, $a_{82} = -370$ and the 6^{th} term, $a_6 = 10$. Find the 33^{rd} term a_{33}.

4. For a given geometric sequence, the 7^{th} term, $a_7 = \frac{23}{25}$ and the 10^{th} term, $a_{10} = 115$. Find the 14^{th} term a_{14}.