3.5 Introduction to Matrices and Gauss Elimination Method

We saw in the previous section how to solve a system of equations. The first thing to notice was for a system of linear equations we created equivalent systems of equations by performing what are called elementary row operations which is basically allowing you to multiply or divide one of the equations in the system by a constant, or add two or more equations together creating a new equivalent system of equations. Two equivalent system of equations have the same solutions.

When we write a collection of numbers into rows and columns they are referred to as a matrix. For example

1. $\begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$ is a 2 by 2matrix (with 2 rows and 2 columns).
2. $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 4 \\ 7 & -1 & 2 \end{bmatrix}$ is a 3 by 3 matrix (with 3 rows and 3 columns)
3. $\begin{bmatrix} -1 & 0 & 4 \\ 3 & -2 & 5 \end{bmatrix}$ is a 2 by 3 matrix (with 2 rows and 3 columns).
4. $\left[\begin{array}{ccc|c} 2 & 3 & 4 & 6 \\ -1 & 3 & 5 & -2 \\ -2 & 1 & 5 & 1 \end{array}\right]$ this matrix is called an augmented matrix and is a 3 by 4 matrix. Such matrices are used to represent system of equations. This particular matrix could represent the system of equations given by $\begin{cases} 2x + 3y + 4z = 6 \\ -x + 3y + 5z = -2 \\ -2x + y + 5z = 1 \end{cases}$ as you can see the coefficients of x, y, and z are in the first, second, third columns respectively and the constant term are in the last column. The equality symbol is replaced with the bar between the third and the fourth column.

We can use matrices to solve very complex systems of equations. For now we will only focus on how to use the elementary row operations to solve system of equations of three or more variables.

The Gauss Elimination Method is basically taking the augmented matrix of a system of linear equations and using elementary row operations bring the matrix into a triangular form that has ones in the diagonals and zeros under them. For example, for a system of linear equations in three variables to get it in the form $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$ this will allow us to get the solutions to the system of equations easily since here for example we will get $\begin{cases} x - y + 2z = 4 \\ y - 3z = -1 \\ z = 2 \end{cases}$ and then we can proceed to get the solutions to system of equations.

Let us see why this works. We will use a two by system of linear equations and then show you the beauty of this approach as we can then solve system of linear equations that has three or more variables. We will represent rows by writing R_n to mean nth row. Elementary row operations will be denoted as $R2 = -3R1 + 2R2$ means the new row 2 is found by adding -3 times row1 to 2 times old row2.

Pay careful attention to the solving of the system of linear equations in two variables and the equivalent step by step use of matrices to solve the same system. This careful attention will pay dividends later to solve more complex system of linear equations.

Practice Examples

Solve the system of equations below.

Solving by elimination method	Solving using matrices
1. $\begin{cases} 3x - 2y = 6 \\ 2x - 5y = 7 \end{cases}$ So first multiply the first equation by 2 and second equation by -3 creating the equivalent system of $\begin{cases} 6x - 4y = 12 \\ -6x + 15y = -21 \end{cases}$ Add the two equations $11y = -9$ or $y = \frac{-9}{11}$ Use this value of y into one of the equations to get the value of x. So we get $3x - 2\left(\frac{-9}{11}\right) = 6$ or $3x = 6 - \frac{18}{11} = \frac{66-18}{11} = \frac{48}{11}$ Or $x = \frac{1}{3}\left(\frac{48}{11}\right) = \frac{16}{11}$. The solution of the system is $x = \frac{16}{11}, y = -\frac{9}{11}$	**1.** $\begin{cases} 3x - 2y = 6 \\ 2x - 5y = 7 \end{cases}$ The augmented matrix to represent this system of equations will be $\left[\begin{array}{cc\|c} 3 & -2 & 6 \\ 2 & -5 & 7 \end{array}\right]$ $\xrightarrow[R2=-3R2]{R1=2R1} \left[\begin{array}{cc\|c} 6 & -4 & 12 \\ -6 & 15 & -21 \end{array}\right]$ $\xrightarrow{R2=R2+R1} \left[\begin{array}{cc\|c} 6 & -4 & 12 \\ 0 & 11 & -9 \end{array}\right]$ $\xrightarrow{R2=\frac{1}{11}R2} \left[\begin{array}{cc\|c} 6 & -4 & 12 \\ 0 & 1 & -\frac{9}{11} \end{array}\right]$ $\xrightarrow{R1=4R2+R1} \left[\begin{array}{cc\|c} 6 & 0 & 12 - \frac{36}{11} \\ 0 & 1 & -\frac{9}{11} \end{array}\right]$ $\xrightarrow{R1=\frac{1}{6}R1} \left[\begin{array}{cc\|c} 1 & 0 & 2 - \frac{6}{11} \\ 0 & 1 & -\frac{9}{11} \end{array}\right] = \left[\begin{array}{cc\|c} 1 & 0 & \frac{16}{11} \\ 0 & 1 & -\frac{9}{11} \end{array}\right]$ $x = \frac{16}{11}$ and $y = -\frac{9}{11}$

It may look like a lot of work compared to use matrices but when working with more than two variables you will see the efficiency of it. So let's attempt to solve a three by three system of linear equations.

2. $\begin{cases} 2x - 3y + z = 7 \\ 3x + 2y - 2z = -3 \\ -x + y + 3z = 4 \end{cases}$

Our goal is to get ones in the diagonals and zeros under it so we can start with interchanging rows one and three represented as below. All that means is we are writing the third equation first and the first equation third. Interchanging equations creates an equivalent system of equations. Then continue with the elementary row operations.

Interchange R1 and R3

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 7 \\ 3 & 2 & -2 & -3 \\ -1 & 1 & 3 & 4 \end{array}\right] \xrightarrow{R1 \leftrightarrow R3} \left[\begin{array}{ccc|c} -1 & 1 & 3 & 4 \\ 3 & 2 & -2 & -3 \\ 2 & -3 & 1 & 7 \end{array}\right] \xrightarrow{R1 = -R1} \left[\begin{array}{ccc|c} 1 & -1 & -3 & -4 \\ 3 & 2 & -2 & -3 \\ 2 & -3 & 1 & 7 \end{array}\right]$$

$R2 = R2 - 3R1$

$$\begin{array}{rrrr} 3 & 2 & -2 & -3 \\ +(-3 & 3 & 9 & 12) \\ \hline 0 & 5 & 7 & 9 \end{array}$$

$R3 = R3 - 2R1$

$$\begin{array}{rrrr} 2 & -3 & 1 & 7 \\ +(-2 & 2 & 6 & 8) \\ \hline 0 & -1 & 7 & 15 \end{array}$$

Interchange R2 and R3

$$\xrightarrow[R3 = R3 - 2R1]{R2 = R2 - 3R1} \left[\begin{array}{ccc|c} 1 & -1 & -3 & -4 \\ 0 & 5 & 7 & 9 \\ 0 & -1 & 7 & 15 \end{array}\right] \xrightarrow{R2 \leftrightarrow R3} \left[\begin{array}{ccc|c} 1 & -1 & -3 & -4 \\ 0 & -1 & 7 & 15 \\ 0 & 5 & 7 & 9 \end{array}\right]$$

$R3 = R3 - 5R2$

$$\begin{array}{rrrr} 0 & 5 & 7 & 9 \\ +(0 & -5 & 35 & 75) \\ \hline 0 & 0 & 42 & 84 \end{array}$$

$$\xrightarrow{R2 = -1(R2)} \left[\begin{array}{ccc|c} 1 & -1 & -3 & -4 \\ 0 & 1 & -7 & -15 \\ 0 & 5 & 7 & 9 \end{array}\right] \xrightarrow[R1 = R1 + R2]{R3 = R3 - 5R2} \left[\begin{array}{ccc|c} 1 & 0 & -10 & -19 \\ 0 & 1 & -7 & -15 \\ 0 & 0 & 42 & 84 \end{array}\right]$$

$R1 = R1 + 10R3$

$$\begin{array}{rrrr} 1 & 0 & -10 & -19 \\ +(\ 0 & 0 & 10 & 20) \\ \hline 0 & 0 & 0 & 1 \end{array}$$

$$\xrightarrow{R3 = \frac{1}{42}R32} \left[\begin{array}{ccc|c} 1 & 0 & -10 & -19 \\ 0 & 1 & -7 & -15 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[R1 = R1 + 10R3]{R2 = R2 + 7R3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution: $x = 1, y = -1, z = 2$

3. $\begin{cases} x+y-z=2 \\ 2x+2y-2z=4 \\ 3x-2y+z=1 \end{cases}$

Our augmented matrix will be

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & 2 & -2 & 4 \\ 3 & -2 & 1 & 1 \end{array}\right] \xrightarrow[R3=R3-3R1]{R2=R2-2R1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 4 & -5 \end{array}\right] \xrightarrow{R2\leftrightarrow R3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -5 & 4 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R2=\frac{1}{-5}R2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -4/5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R1=R1-R2} \left[\begin{array}{ccc|c} -1 & 0 & -1/5 & 1 \\ 0 & 1 & -4/5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the bottow row is always true we can set $z=t$ where $t = any\ real\ number$

That makes $y=\frac{4}{5}t+1$ and $x=\frac{1}{5}t+1$.

There are infinitely many solutions to this sytem.

4. $\begin{cases} x+y-z=2 \\ 2x+2y-2z=6 \\ -x-2y+z=-3 \end{cases}$

Our augmented matrix will be

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & 2 & -2 & 6 \\ -1 & -2 & 1 & -3 \end{array}\right] \xrightarrow[R3=R3+R1]{R2=R2-2R1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & -1 & 0 & -1 \end{array}\right]$$

In row 2 we notice the equation $0=2$ which is never true so this system of equations has no solutions.

Cover Sheet for Sec 3.5 Name: ______________________________

1. Why should we bother learning about the Gauss Elimination method?

2. What are the advantages of writing the system of equations using matrices?

3. Do you think we use matrices for a two by two system of linear equations?

4. What about using matrices for two by two nonlinear system of equations?

Video Log 3.5

Solve the following systems using Gauss Elimination Method.

1. $\begin{cases} x - 2y + z = 4 \\ 3x - y + 6z = -1 \\ 4x + 3y - 7z = 5 \end{cases}$

2. $\begin{cases} 4x + 5y + 2z = -5 \\ x + y - 3z = -4 \\ -3x + 2y - z = -13 \end{cases}$

3. $\begin{cases} x + y + 2z = 4 \\ 2x + 2y + 4z = -4 \\ -3x + 2y - z = -13 \end{cases}$

4. $\begin{cases} x - y + 2z = -5 \\ x + y - 3z = -4 \\ -2x + 2y - 4z = 10 \end{cases}$

5. $\begin{cases} x + y + z + w = 1 \\ 2x - y + 3z - y = -9 \\ x - y - z + w = 1 \\ x - 2y + z - 2w = -8 \end{cases}$