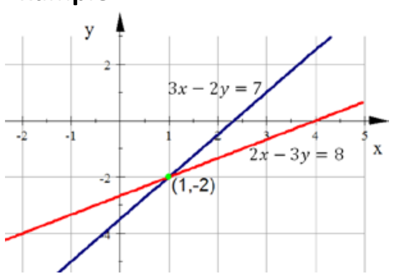
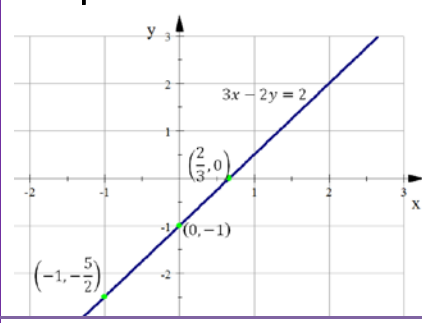
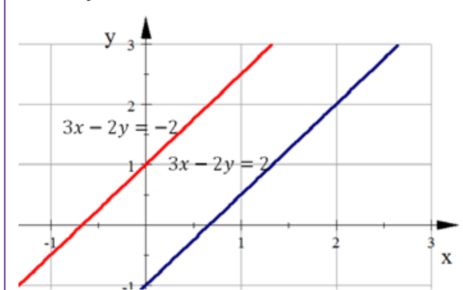
3.4 Systems of Equations

We know how to solve some types of equations in one variable. But very often there is a need to solve a collection of more than one equation in two or more variables. So by now you are familiar on how a mathematician uses prior knowledge to play with the objects at hand to deliver the solutions. So let us look at certain types of systems of equations which is a collection of equations in two or more variables and see how we could find the solutions if they exist.

A system of linear equations in two variables or more variables is a set of two or more linear equations. A system of nonlinear equations in two variables or more variables is a set of two or more nonlinear equations. A solution to a system of equations is a point in the plane that satisfies all the equations in the system.

Recall when working with a system of linear equations in two variables means working with lines. If two lines intersect each other they are said to form a consistent independent system of equations with a unique solution, which is the point of intersection. If two lines overlap each other, or are the same line, then all points on the line are solutions to the system of equations. Such a system of equations is called a consistent dependent system of equations with infinitely many solutions. If two lines are parallel to each other they are said to form an inconsistent system of equations with no solution. Graphically, the three scenarios are depicted below.

Consistent Independent	Consistent Dependent	Inconsistent
Unique Solution	Infinitely many solutions	No Solutions
Example 	**Example** 	**Example** 
The two lines are not parallel and therefore intersect in one point. The point of intersection will be the unique solution to the system of equations.	The lines are parallel and have the same y-intercept or the lines overlap. In this situation all points on the line are solutions to the system of equations and therefore has infinitely many solutions.	The lines are parallel and have different y-intercepts. In this situation no point in the plane can be the solution and therefore the system has no solutions.

When working with a system of nonlinear equations, knowing what shape the graphs of each of the equations look like may also help in understanding the nature of the solutions or how many to expect.

Methods to solve a system of equations and inequalities

Graphical Method: Plot the graphs of all the equations or inequalities in the system on the same coordinate system and then from the graph locate the solution or the solution set if one exists. Depending on the graph though you may at times only be able to get an **approximate value** of the x and y coordinates of the solution or the point that satisfies all the equations, or inequalities in the system.

Graphing method is not always the best can give you an insight into the solutions.

Substitution Method: Solve one of the equations for either the x or y and substitute this value into the other equation making it then an equation in one variable. This will allow you to solve the new equation in one variable. Then replace that value in one of the original equations to find the other coordinate.

Elimination Method: Align the equations one above the other so that all the variables and degrees of each of the equations are lined up. Then you multiply one of both of the equations by the required constants so that when you add the two equations it results in a one variable equation and you can then proceed as in the substitution method.

When dealing with more than two variables the first two methods will work, the elimination method becomes a little more complicated but can still come in handy. In this case we will see what our options are later.

Practice examples

Solve the systems of equations if possible.

1. $\begin{cases} 3x - y = 2 \\ 2x + y = 3 \end{cases}$

Substitution Method

Solve the first equation for y $y = 3x - 2$

Substitute this value of y in the second equation

$2x + 3x - 2 = 3$ or $5x = 5$, or $x = 1$

$$y = 3(1) - 2 = 3 - 2 = 1$$

Solution: $(1,1)$

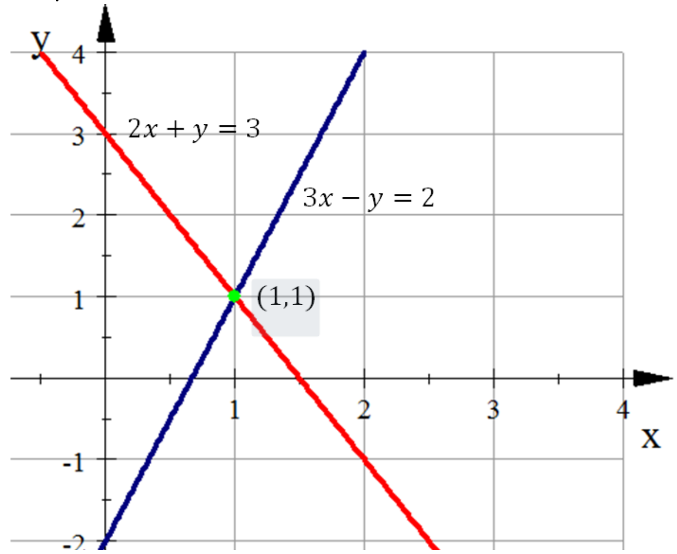
Elimination Method

Add the two equations and we get

$$5x = 5$$
$$x = 1$$
$$3(1) - y = 2$$
$$y = 1$$

Solution: $(1,1)$

Graphical Method



2. $\begin{cases} 2x + y = 2 \\ 4x + 2y = 3 \end{cases}$

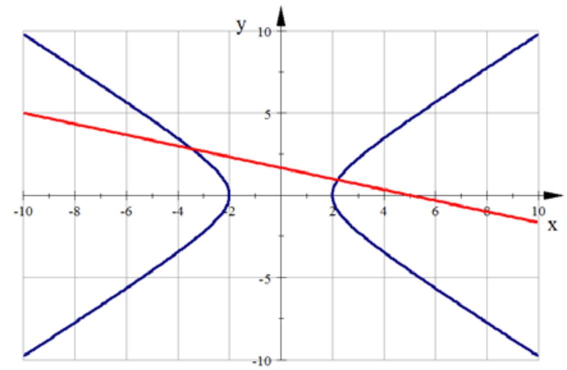
Elimination Method

Multiply the first equation by -2

$\begin{cases} -4x - 2y = -4 \\ \ \ 4x + 2y = 3 \end{cases}$ add the two equations and we get

$0 = -1$ a false statement therefore this system of equations has no solution or is an inconsistent system.

3. $\begin{cases} x^2 - y^2 = 4 \\ x + 3y = 5 \end{cases}$



We can see from our knowledge of graphs that the first equation is a hyperbola and the second a line so we expect two solutions. As you can see it is not easy to read the solutions from their graphs but we got an insight into how many solutions to expect. So here substitution method will come in handy.

Solve the second equation for x and we get $x = -3y + 5$. Now substitute this value in the first equation we get

$$(-3y + 5)^2 - y^2 = 4$$
$$9y^2 - 30y + 25 - y^2 = 4$$
$$8y^2 - 30y + 21 = 0$$

Using quadratic formula we get

$$y = \frac{30 \pm \sqrt{(-30)^2 - 4(8)(21)}}{16} = \frac{15 \pm \sqrt{57}}{8}$$

$y = \frac{15}{8} + \frac{\sqrt{57}}{8}$ will give us $x = -3\left(\frac{15}{8} + \frac{\sqrt{57}}{8}\right) + 5$ Or $x = -\frac{5}{8} - \frac{3\sqrt{57}}{8}$

$y = \frac{15}{8} - \frac{\sqrt{57}}{8}$ will give us $x = -3\left(\frac{15}{8} - \frac{\sqrt{57}}{8}\right) + 5$ Or $x = -\frac{5}{8} + \frac{3\sqrt{57}}{8}$

Solutions are

$\left(-\frac{5}{8} - \frac{3\sqrt{57}}{8}, \frac{15}{8} + \frac{\sqrt{57}}{8}\right)$ and $\left(-\frac{5}{8} + \frac{3\sqrt{57}}{8}, \frac{15}{8} - \frac{\sqrt{57}}{8}\right)$

4. $\begin{cases} x^2 + 2y = 4 \\ x - 2y = -2 \end{cases}$

Elimination Method

Adding the two equations we get

$x^2 + x = 2$ or $x^2 + x - 2 = 0$

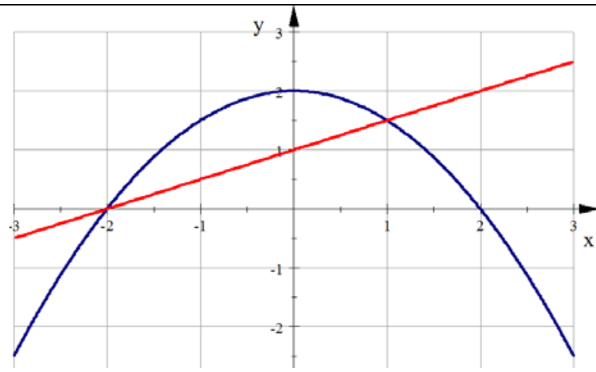
$$(x - 1)(x + 2) = 0$$

$x = 1$ or $x = -2$

$x = 1$ gives us $1 - 2y = -2$ or $y = \frac{3}{2}$

$x = -2$ gives us $-2 - 2y = -2$ or $y = 0$

Solutions are $(-2,0)$ and $\left(1, \frac{3}{2}\right)$.



Graphing does give us solutions here as $(-2,0)$ and $\left(1, \frac{3}{2}\right)$.

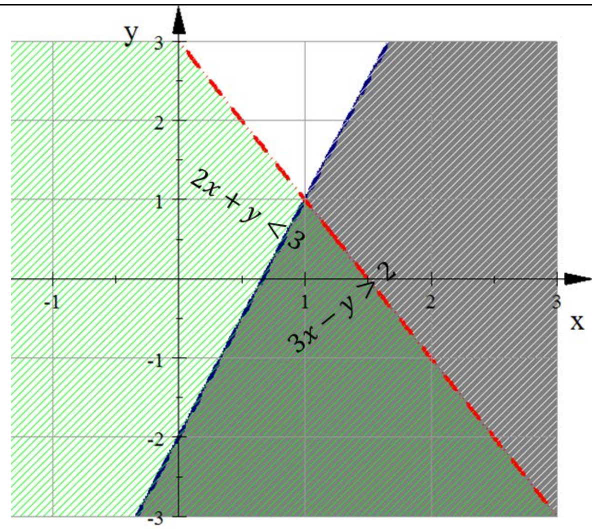
5. $\begin{cases} 3x - y > 2 \\ 2x + y < 3 \end{cases}$

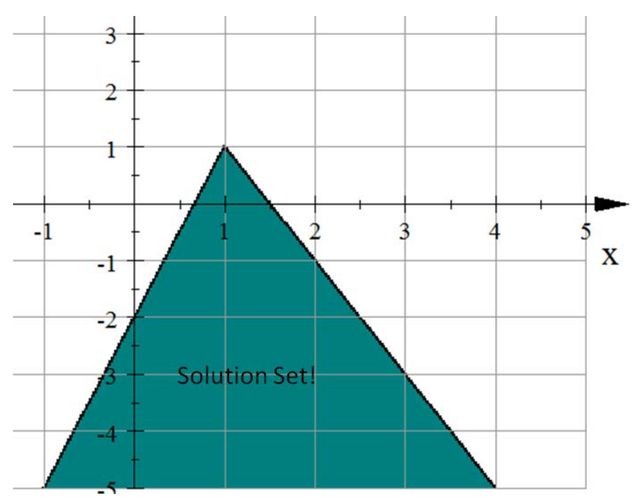
Step 1: Plot each of the line. Determine whether each of the lines is dotted or solid based (dotted if working with a inequality, and solid if working with either $\leq$, or $\geq$).

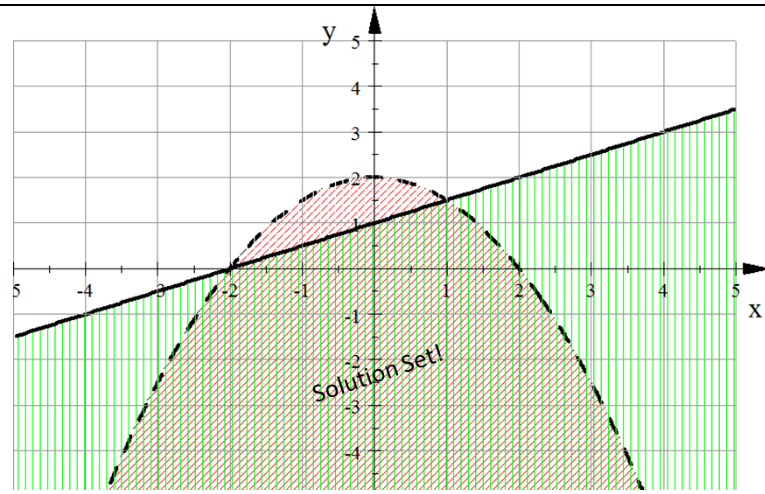
Step 2: Take a test point on the lines to see if it is part of the solution or not, then shade the appropriate region.

In this case each of the lines are dotted lines since we have strict inequalities.

Let us pick $(0,0)$ as our test point. In the first inequality we see $0 > 2$ is a false statement so all the points in the region $3x - y = 2$ are solutions to this inequality. For the other inequality $(0,0)$ as our test point we see $0 < 3$ which is true so all points in the place above the line $2x + y$.





6. $\begin{cases} x^2 + 2y < 4 \\ x - 2y \geq -2 \end{cases}$ Plot each inequality and the overlap is the solution as you see in the picture to the right.	

Cover Sheet for Sec 3.4 Name: ______________________________

1. What is a system of equations?

2. When would you use the graphing method?

3. When would you use the elimination method?

4. When would you use the substitution method?

5. What is the difference between solving system of equations versus system of inequalities?

6. What is most number of solutions you expect from of a system of equations where both equations are second degree polynomials?

7. What is least number of solutions you expect from of a system of equations where both equations are second degree polynomials?

Video log 3.4

Find the solutions to system of equations and inequalities below.

1. $\begin{cases} 5x - y = 4 \\ x - 2y = 3 \end{cases}$

2. $\begin{cases} \frac{1}{2}x - y = 4 \\ x - 2y = 3 \end{cases}$

3. $\begin{cases} x - 2y = 4 \\ -4x + 16 = -8y \end{cases}$

4. $\begin{cases} x^2 + y^2 = 4 \\ x - y^2 = -2 \end{cases}$

5. $\begin{cases} x^2 + y^2 = 4 \\ x^2 - y^2 = 1 \end{cases}$

6. $\begin{cases} x^2 + y^2 = 4 \\ x - y^2 = -6 \end{cases}$

7. $\begin{cases} 9x^2 + 4y^2 = 36 \\ x = 3 \end{cases}$

8. $\begin{cases} 9x^2 + 4y^2 = 36 \\ y = 3x - 1 \end{cases}$

9. $\begin{cases} 9x^2 + 4y^2 = 36 \\ 3x - y = -12 \end{cases}$

10. $\begin{cases} x - y < 4 \\ x - 2y \geq 3 \end{cases}$

11. $\begin{cases} x^2 + y^2 \leq 4 \\ x - y^2 < 1 \end{cases}$

12. $\begin{cases} x^2 + y^2 > 4 \\ x^2 - y^2 < 1 \end{cases}$

13. $\begin{cases} x^2 + y^2 \geq 4 \\ x - y^2 \geq -6 \end{cases}$