Chapter 1: It's All About Functions and Relations

1.1 Introduction and Domain Range of Functions and Relations

Lecture

- Functions and Relations Part 1 (13.15 min)
  https://www.youtube.com/watch?v=2HCxvI_S-WQ

We very often consider how different things are related in many situations in every-day life. Some examples include: each person is related to their biological mother; the price of gas in $\frac{\$}{gal}$ is related to time; temperature in Fahrenheit degrees is related to temperature in Celsius degrees; your height in inches is related to your age in years; the profit of an automaker is related to the annual car sales; the number of movie ticket sales at a theater is related to the ticket price. Many of these relationships can be quantified using numerical measures of each related quantity, e.g. the height of a person in inches on his birthday is related to his age in years, or the relationship between Fahrenheit and Celsius temperatures is given by the equation $°\mathrm{F} = \frac{9}{5}°\mathrm{C} + 32$ for any value of $°\mathrm{C}$.

The connections between different quantities can be referred to as **relations**.

Oftentimes we think of a direction in the relationship in that when the value of one quantity (called an input) is known, this determines a value of the second quantity (called an output).

A **function** is a relation where every individual input has a unique output.

The word relation means two or more items are related. We will focus on relations between two items or characteristics that are connected to each other. For example, think of the students in a class and their height in inches. We could represent these data as a set of ordered pairs or as a table.

A={(Robert,72),(Sarah,61), (Matt,70), (Robin,61),(Sarah, 65)}

Name of Student	Height in Inches
Robert	72
Sarah	61
Matt	70
Robin	61
Sarah	65

In the data pairs above, two people could have different names and have the same height. Also two people can have the same name but different heights, or two could have the same height and the same name with a larger class. The way this data is presented suggests that the name of the person is the "input", while the height of that person is the "output".

Clearly the relation above is not a function, since the single input "Sarah" has two different outputs 61'' and 65''. Note that if the input column included middle initial, e.g., B={(Robert A, 72),(Sarah K, 61), (Matt L, 70), (Robin P , 61),(Sarah M, 65)}, then each input has a single output and B would be a functional relation of the form $(name, height)$.   However the relation B in the other direction from height to name would not be a function since in that direction, the input of 61" yields two different names!

Consider the sister relationship where any person with one or more sisters is the input and that person's sister(s) is the output. Since there are people who have more than one sister, this is not a function relationship. Next, consider the biological mother relationship where any person is an input and the output is that person's biological mother. Explain why this is a function. Note if we reverse the direction of this Mother relation, it is not a function, i.e., starting with an input of a mother with more than one child, that mother would give rise to at least two different outputs (her children).

Another way to express some relations and functions is through an equation with two variables. For example, $y = 3x + 1$, where the input x is any real number, and y is the corresponding output number. This is a relation between x and y that is a function since for any value of the input x, there is only one output, namely $y = 3x + 1$. We'd say this equation represents a function from x to y.   In the relation $y^2 = x$, where input x is any nonnegative real number, y is not a function of x, since the input of $x = 4$ will yield two different outputs $y = 2$, $y = -2$.

In order to distinguish two different functions like $y = 3x + 1$ and $y = x - 5$, we need a notation that clearly describes the two "y"s as being different. We use the notation $f(x) = 3x + 1$ and $g(x) = x - 5$ where we are giving names of f and g to these two different functional relations . We would read that as "f of x is $3x + 1$" and "g of x is $x - 5$".   We could also write this as e.g., $y = f(x) = 3x + 1$ which identifies the function equation $y = 3x + 1$ with the function name f. It is very important to make sure you learn to read and process this notation correctly. Many students who make mistakes in working with functions confuse the notation $f(x)$ with f times x. Remember that the notation $f(x)$ is NOT read as f times x. You can think of it as $f($ ___ $)$, and the blank can be filled in with many different inputs.

In the notation $f(x) = 3x + 1$, f is called the name of the function, and x is called the input or argument, $f(x)$ refers to the output. This statement simply says that the function f computes its outputs by multiplying an input by 3 and then adding 1.

The collection of all the input x values we can have in a function so that the output is well defined is called the **domain** of the function. The collection of all the output values of a function is called the **range** of the function.

Working with Domain and Range of a Function or a Relation

Playing

Remember there cannot be a function or a relation without having a domain and range. In mathematics we are always trying to use our imagination to see what all possibilities are in every new scenario.

Some questions to ask are

1. What kinds of sets of objects we can be used for domains and ranges.
2. Do functions have to have one input and one output, or could we have multiple inputs and one unique output?

So before you continue reading more spend some time being creative in answering these questions above yourself. Remember there really are no rules here. Try being creative and see what you can come up with.

When a function has one input and one output it is called a function of one variable. When a function has multiple inputs and one output we call it a function of several variables. The domain of a function can be anything you want it to be from finite set of objects to infinite sets. We will mostly work with functions of single variable in this book. We need to develop some new notation and convention on how work with functions to become more efficient. Any letter really can represent the single input. For example, we may use letters like $x, or\ t$ to represent an input to a function. Similarly we can really use some different letters to represent the corresponding single output. For example letters like $y, or\ z$ may represent the output to a function. The functional relationship itself will often be denoted by the letter f or some other appropriate letter. A statement like $y = f(x)$ simply says that y represents an output, x an input and they are related by the function called f. (Remember that this is not saying to multiply f by x!) To define what this relationship is, we'd have to elaborate on the meaning of y and x and how y and x are related perhaps by some equation. We can also look at the function notation as follows where the object x is transformed by the application of the function f to become a new object called $f(x)$ (read as f of x)

$$x \xrightarrow{f} f(x)$$

This notation is credited to the mathematician Leonard Euler and was developed around 1734.

To denote what kind of function we are working with mathematicians may use the notation shown below. Writing $f: \mathbb{R} \rightarrow \mathbb{R}$ to means we have a real number as an input and output is a real number. Such a function is called a real valued function of one variable. Writing $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ to mean the function f is taking an ordered pair of real numbers and transforming them into a real number. Such a function is called a real valued function of two variables. We can take as many variables as needed. So we do not have to stop at a function of two variables. Writing $f: \mathbb{R}^n \rightarrow \mathbb{R}$ to means we have an n-tuple as an input and output is a real number.

We can represent functions in many ways. Below are some of the ways.

a) Using function notation as described above.
   1 $f(x) = x^2$
     Here the domain is set of all real numbers and range is set of all non-negative real numbers.
   2 $f(x, y) = x^2 + y^2$
     Here the domain is the set of tuples of the type (x, y) with x and y both real numbers and the range is the set of all non-negative real numbers.
   3 $f(t) = 2t - 5$
     Here the domain is set of all real numbers and range is the set of all real numbers.
b) As a set of objects. For example,
   1. $f: \{(2, -1), (4,2), (3,2), (0,4)\}$.

Read this is as when input is 2 output is -1, input is 0 output is 4 and so on. Here the domain is the set $\{2,4,3,0\}$ and the range is the set $\{-1,2,4\}$ as you can see we did not write the number 2 in the range twice.

2. $f\colon\{(x,y)|y = x^2, x \in \mathrm{R}\}$. Read this as f is the function that is a collection of all ordered tuples of the type (x, y) in which $y = x^2$ and x is a real number. Here the domain is the set of all real numbers and range is set of all non-negative real numbers.

c) As formulas or equations in two or more variables. For example,

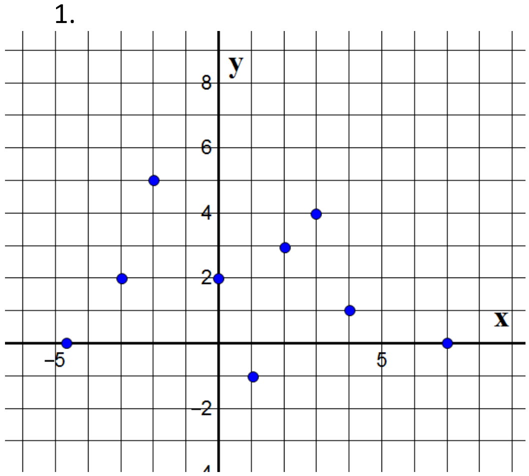
1. $y = x^2$
   Here the domain is the set of all real numbers and range is set of all non-negative real numbers.
2. $z = x^2 + y^2$ Here the domain is the set of tuples of the type (x, y) with x and y both real numbers and the range is the set of all non-negative real numbers.

d) As English sentences modeling scenarios. For example,

1. Linda's parents loaned her 12,480 dollars interest free for her college tuition and books. Linda promised to pay her parents back 80 dollars a week until the loan is completely paid off. Write an equation that will allow Linda a quick overview of how much money she owes her parents in a particular week before it is all paid off.
   The equation would look like Amount owed by Linda
   $12480 - 80 \times (number\ of\ weeks)$
   Here domain is all whole numbers less than or equal to $\frac{12480}{80} = 156$ and range is all whole between zero and 12480 which are multiples of 80.
2. The period of a pendulum is proportional to the square root of its length.
   Here domain is set of all non-zero real numbers and range is also set of non-zero real numbers.

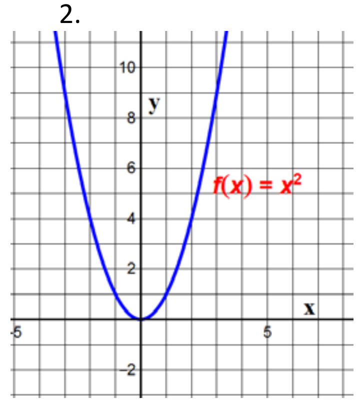
e) As graphs.

1.



Input is the x coordinate of the point and output is the y coordinate of the point. For example input of-2 will give you output of 5 since $(-2,5)$ is a point plotted above.
Here domain is $\{-5,-3,-2,0,1,2,3,4,7\}$ and range is $\{-1,0,1,2,3,4,5\}$.

2.



Input is the x coordinate of a point on the graph of the function $f(x) = x^2$, then output is the y coordinate of that point. For example input of-2 will give you output of 4 since $(-2,4)$ is a point on the graph above. Here the domain is the set of all real numbers and range is set of all non-negative real numbers.

Below are more examples of functions.

1. $°\mathrm{F} = T(°\mathrm{C}) = \frac{9}{5}°\mathrm{C} + 32$. This statement is a description of the $°\mathrm{C} \to °\mathrm{F}$ conversion function. The input is the temperature in °C, the output is the temperature in °F and the name of the function is T. In this course our functions usually will be defined in this way, i.e. $y = f(x) =$(some algebraic formula involving x). Also note that a function is the relation between input and output and that $f(x) = \frac{9}{5}x + 32$ is the same function as $T(°\mathrm{C}) = \frac{9}{5}°\mathrm{C} + 32$ since they have all the same ordered pairs.

2. $V = \frac{4}{3}\pi r^3$ is the volume formula when the radius of a ball is known. This is a polynomial function. We might call this function simply V in which case we'd have $V = V(r) = \frac{4}{3}\pi r^3$ where we use the same variable to denote an output and the name of the relationship. Likewise, the *area of a circle* function can be expressed as $A = A(r) = \pi r^2$.

3. The height of a baseball might be given by $y = h(t) = 2 + 160t - 16t^2$ where y is the height above the ground in feet and t is the number of seconds after the ball is hit by the bat and h is the name of this relationship between time and height. This is also an example of a polynomial function. This notation allows us to easily describe the height at several different times, e.g. $h(2) = 258$ says that when the input is 2 seconds, the height is 258 feet. Other questions can also be stated easily using this notation, e.g., "determine when $h(t) = 100$" says that the output of the height function is 100 and we are to find the input t when this happens. You could try guessing and checking to find out that this happens at $t \approx 0.66 sec, and\ t \approx 9.34\ sec$. You can use the quadratic formula to find exact answers if you already know how to do this, otherwise we will review it for you in a later section.
4. Another type of function that is often useful is called an exponential function where the input is actually in the exponent. For example $P(t) = 7.4(1.012)^t$ might be used as a model of the world's population t years after 2016 where the output is in billions of people. Thus $P(10) = 7.4(1.012)^{10} \approx 8.34\ billion$. This number 8.34 billion is the projection of the world population in 2026.

Playing with function notation

If we say $f(x) = 3x + 1$, then that means that we really have

$f(\underline{\quad\quad}) = 3 \times (\underline{\quad\quad}) + 1$

That means if you are asked to evaluate the following you must replace the blank above with whatever takes its place in the notation

$f(\underline{\quad\quad}) = 3 \times (\underline{\quad\quad}) + 1$

$f(COLD) = 3 \times (COLD) + 1$

$f(2) = 3 \times (2) + 1 = 6 + 1 = 7$

$f(100) = 3 \times (100) + 1 = 300 + 1 = 301$

$f(a) = 3(a) + 1 = 3a + 1$

$f(a + h) = 3(a + h) + 1$

See if you can use these examples to work on the following practice problems.

Practice Problems

1. Let $f(x) = 7x^2 + 1$ and $g(t) = \sqrt{t} + 7$. Find the values of
   a. $f(4)$
   b. $f(a + h)$
   c. $f(-2)$
   d. $g(a + 1)$
   e. $g(4)$
   f. $f(3) + g(a)$
2. Let $Absolute(t) = |t|$. Find the values of
   a. $Absolute(-5.1)$
   b. $Absolute(10)$
   c. $Absolute(-1200)$
   d. $Absolute(1400)$

Solutions

1. Let $f(x) = 7x^2 + 1$ and $g(t) = \sqrt{t} + 7$. Find the values of
   a. $f(4) = 7 \cdot 4^2 + 1 = 7 \cdot 16 + 1 = 113$
   b. $f(a + h) = 7 \cdot (a + h)^2 + 1$
   c. $f(-2) = 7 \cdot (-2)^2 + 1 = 7 \cdot 4 + 1 = 29$
   d. $g(a + 1) = \sqrt{a + 1} + 7$
   e. $g(4) = \sqrt{4} + 7 = 9$
   f. $f(3) + g(a) = 7 \cdot (3)^2 + 1 + \sqrt{a} + 7 = 63 + 1 + \sqrt{a} + 7 = 71 + \sqrt{a}$
2. Let $Absolute(t) = |t|$. Find the values of
   a. $Absolute(-5.1) = 5.1$

b. $Absolute(10) = 10$

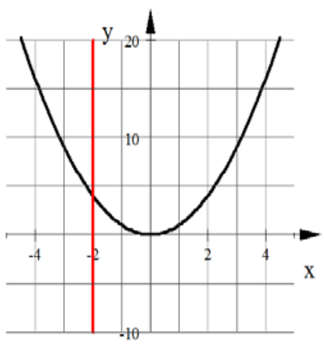
c. $Absolute(-1200) = 1200$

d. $Absolute(1400) = 1400$

Vertical Line Test:
The graph of a relation is a function if and only if any vertical line intersects the graph in one or less points.

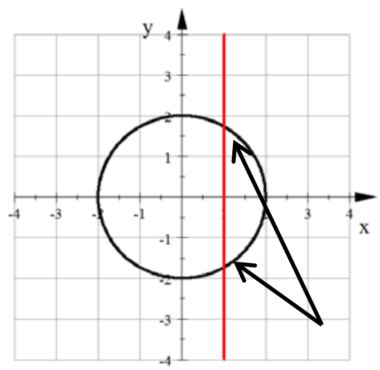
Examples

1. Below y **<u>is</u>** a function of x as it passes the vertical line test.



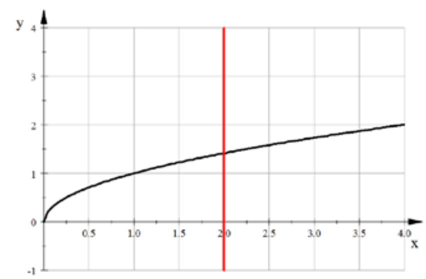
As you can see from the graph above any vertical line will intersect the graph in only one point.

2. Below y **<u>is not</u>** function of x as it does not pass the vertical line test.



As you can see from the graph above the vertical line intersects the graph in two points.

3. Below y **<u>is</u>** a function of x as it passes the vertical line test.



As you can see from the graph above any vertical line will intersect the graph in only one point.

Playing

If you look at examples 1 and 3 above, a natural question that we can ask is what information would a horizontal line test provide? We can see that if a horizontal line is drawn in example 1, we can say x would not be a function of y. Or that in the relation $y = x^2$, x is not a function of y. However, in example 3 we can say x would be a function of y. Or that in the relation $y = \sqrt{x}$, x is a function of y. The relation $y = \sqrt{x}$ can also be thought of as $y^2 = x$, with $y \geq 0$. A function that satisfies the horizontal line test is then called a one-one-to function.

One-to-One Function: A function in which each output has one and only one input is called a <u>one-to-one</u> function.

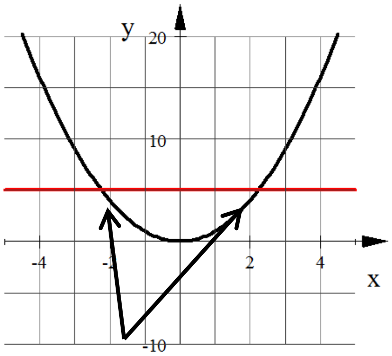
Horizontal Line Test:
A function is one-to-one if and only if any horizontal line intersects the graph in one or less points.

To check algebraically whether a function is one-to-one, we need to make sure that the outputs of a function f are equal, $(a) = f(b)$, only when $a = b$.

Examples

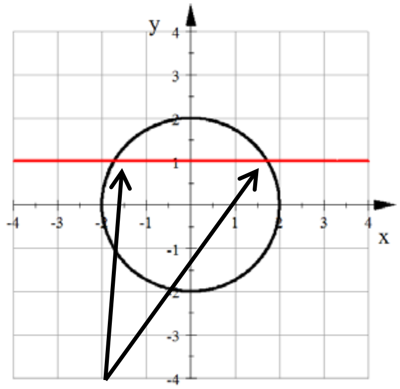
Determine if the functions shown the graph of the equations are one-to-one.

1. Below y **<u>is not</u>** a one-to-one function of x as it does not pass the horizontal line test.



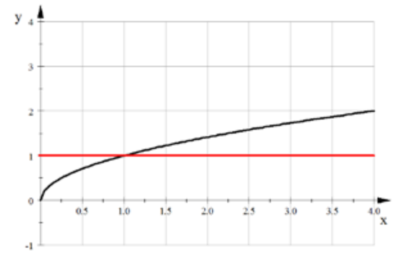
As you can see from the graph above the horizontal line will intersect the graph in two points.

2. Below y **<u>is not</u>** a one-to-one function of x as it does not pass the horizontal line test.



As you can see from the graph above the horizontal line intersects the graph in two points.

3. Below y **<u>is</u>** a one-to-one function of x as it passes the horizontal line test.



As you can see from the graph above any horizontal line will intersect the graph in only one point.

4. $y = 3x - 1$

We could graph the relation and use horizontal line test as above or check algebraically whether the function is one-to-one. If $x = a$, and $x = b$ are two real values, then $3a - 1 = 3b - 1$, solving the equations for b we get $3a = 3b$, or $a = b$. That is the function values are only equal if $a = b$ implying the function is one-to-one.

5. $y^3 = x$

Checking algebraically whether the function is one-to-one: For two real values $y = a$, and $y = b$ If $a^3 = b^3$, solving the equations for b we get $a = b$

That is the function values are only equal if $a = b$ implying the function is one-to-one.

6. $y + 3x^2 = 4$

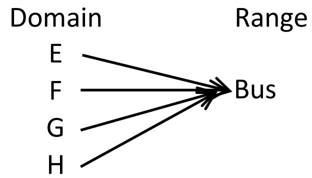
or $y = 4 - 3x^2$

Checking algebraically whether the function is one-to-one: For two real values $x = a$, and $x = b$, $4 - 3a^2 = 4 - 3b^2$, we get $a^2 = b^2$ or that $a = \pm b$. That implies the function is not one-to-one.

Practice Problems

1. Determine the following
   i. Is the relation a function or not.
   ii. Is the relation a one-to-one function
   iii. Domain and Range

A.

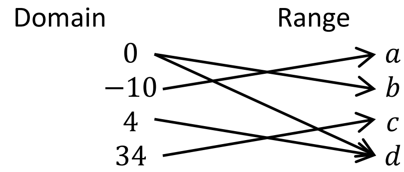


o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

B.



o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

C.

$$\{(a,b),(a,2),(b,2),(c,4)\}$$

o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

D.

$$\{(a,-1),(b,2),(c,2),(d,4)\}$$

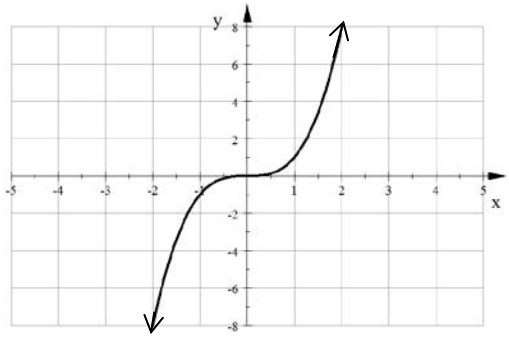
o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

E. An Olympic size swimming pool holds 2,500,000 liters of water. The pool currently holds 100,000 liters of water, and water is being pumped at 400,000 liters/hour into the tank. Write a function that represents the amount of water in the tank at t hours. Find the domain and range of this function.

F.

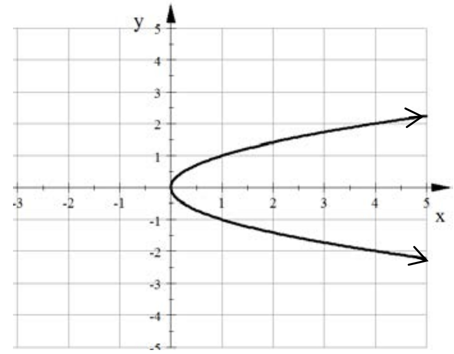


o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

G.



o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

H. $f(x) = |x|$

o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

I. $x^2 + y^2 = 25$

o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

J. $Temp(t) = \frac{5}{9}(t - 32)$

o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

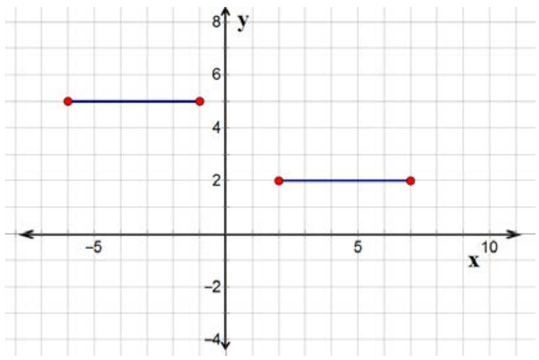
K. $y^3 = x$

o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

L.

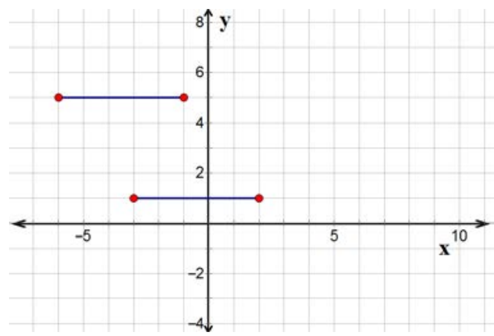


o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

M.

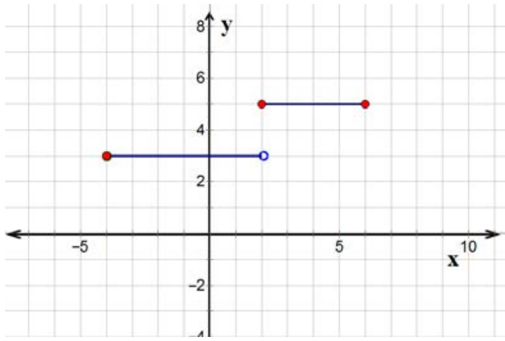


o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

N.

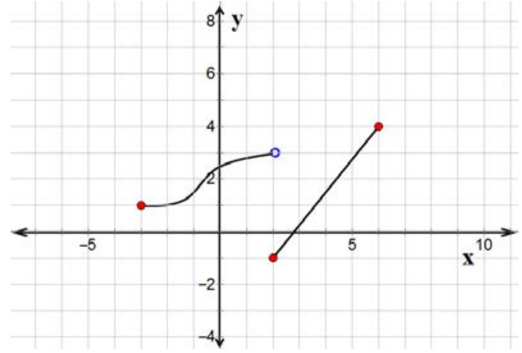


o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

O.



o Function	o One-to-One
o Not a Function	o Not One-to-One

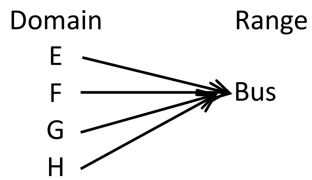
Domain:

Range:

Practice Problems Solutions

1. Determine the following
   i. Is the relation a function or not.
   ii. Is the relation a one-to-one function
   iii. Domain and Range

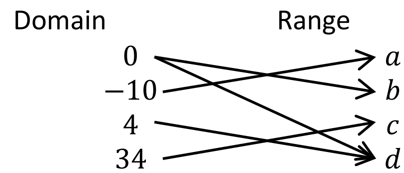
A.



✓ Function	o One-to-One
o Not a Function	✓ Not One-to-One

Domain: {E,F,G,H**}**
Range: {Bus**}**

B.



o Function	o One-to-One
✓ Not a Function	o Not One-to-One

Domain:{$0, -10, 4, 34$**}**
Range: {a, b, c, d**}**

C.

$$\{(a,b),(a,2),(b,2),(c,4)\}$$

o Function	o One-to-One
✓ Not a Function	o Not One-to-One

Domain: {a, b, c**}**
Range: {$b, 2, 4$**}**

D.

$$\{(a,-1),(b,2),(c,2),(d,4)\}$$

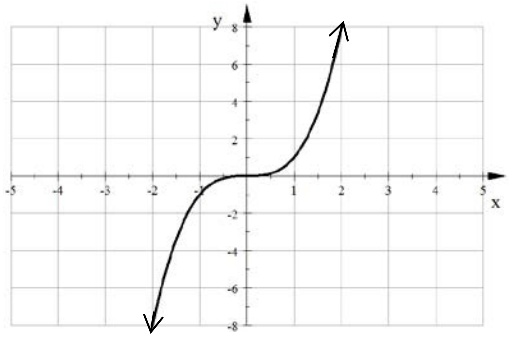
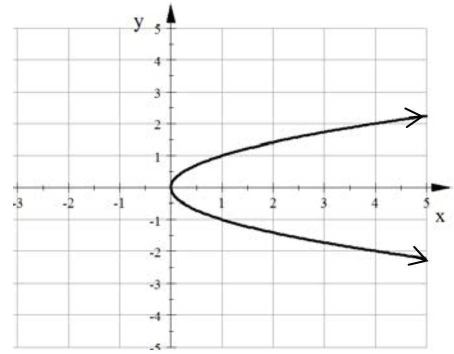
✓ Function	o One-to-One
o Not a Function	✓ Not One-to-One

Domain: {a, b, c, d**}**
Range: {$-1, 2, 4$**}**

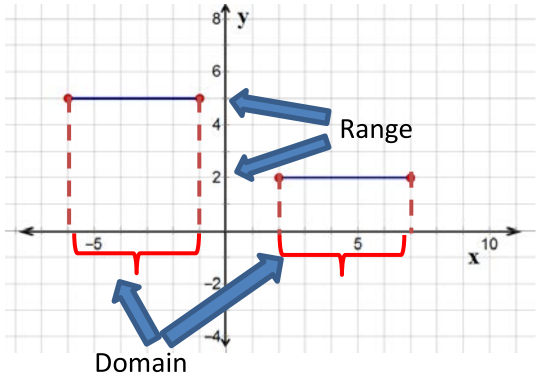
E. An Olympic size swimming pool holds 2,500,000 liters of water. The pool currently holds 100,000 liters of water, and water is being pumped at 400,000 liters/hour into the tank. Write a function that represents the amount of water in the tank at t hours. Find the domain and range of this function.

We will represent the amount of water time t hours as $A(t)$. Since there is already 180000 liters of water in the tank and then we are adding more water per hour we have $A(t) = 100000 + 400000t$.

To find domain and range we have to think about what t values we can have. Note that the water in the tank cannot exceed the pool size of 2,500,000 liters. So we have
$100000 \leq 100000 + 400000t \leq 2500000$ solving the inequality we will get that
$0 \leq 400000t \leq 2500000 - 100000$
$0 \leq 400000t \leq 2400000$ or
$0 \leq t \leq \frac{2400000}{400000}$ or $0 \leq t \leq 6$, also note that when time $t = 6$ hours the pool will be at full capacity of 2500000 and at time $t = 0$ the pool would have 100000 liters.
Domain $= [0,6]$
Range $[100000, 2500000]$

F.  y is a function of x and vice versa. ✓ Function | ✓ One-to-One o Not a Function | o Not One-to-One **Domain:** $(-\infty, \infty)$ **Range:** $(-\infty, \infty)$	G.  Only y is a function of x o Function | o One-to-One ✓ Not a Function | o Not One-to-One **Domain:** $[0,\infty)$ **Range:** $(-\infty, \infty)$
H. $f(x) = \|x\|$ For every real number $x = a$, we have $y = \|a\|$ which results in a unique real number. ✓ Function | o One-to-One o Not a Function | ✓ Not One-to-One **Domain:** $(-\infty, \infty)$ **Range:** $[0,\infty)$	I. $x^2 + y^2 = 25$ For every real number $x = a$, we have $y = \pm\sqrt{25 - a^2}$ which are two real numbers for any a that satisfies $-5 \leq a \leq 5$. o Function | o One-to-One ✓ Not a Function | o Not One-to-One **Domain:** $[-5,5]$ **Range:** $[-5,5]$
J. $Temp(t) = \frac{5}{9}(t - 32)$ For every real number t we will get a unique real number $\frac{5}{9}(t - 32)$. For any two values of $= a$, and $t = b$ we have $\frac{5}{9}(a - 32) = \frac{5}{9}(b - 32)$ or $a - 32 = b - 32$, or $a = b$ which means the function is one-to-one. ✓ Function | ✓ One-to-One o Not a Function | o Not One-to-One **Domain:** $(-\infty, \infty)$ **Range:** $(-\infty, \infty)$	K. $y^3 = x$ y is a function of x and vice versa. For every real number $x = a$ we have one and only one output $y = \sqrt[3]{a}$. For any two values of $= a$, and $y = b$ we $\sqrt[3]{a} = \sqrt[3]{b}$ or that $a = b$ which means the function is one-to-one. ✓ Function | ✓ One-to-One o Not a Function | o Not One-to-One **Domain:**$(-\infty, \infty)$ **Range:** $(-\infty, \infty)$

L.



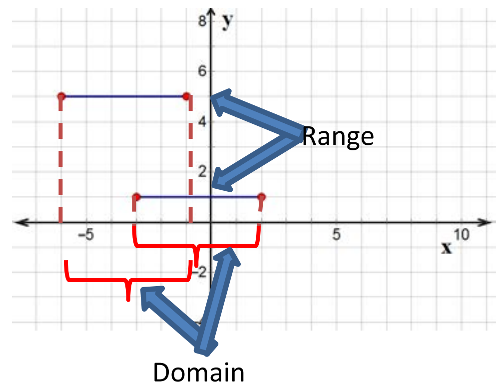
Domain: $[-6,-1] \cup [2,7]$

Range: $\{\mathbf{5},\mathbf{2}\}$

Each x coordinate has a unique y coordinate, so y is a function of x. But several x coordinates have the same output, so y is not a one-to-one function of x.

✓ Function	o One-to-One
o Not a Function	✓ Not One-to-One

M.



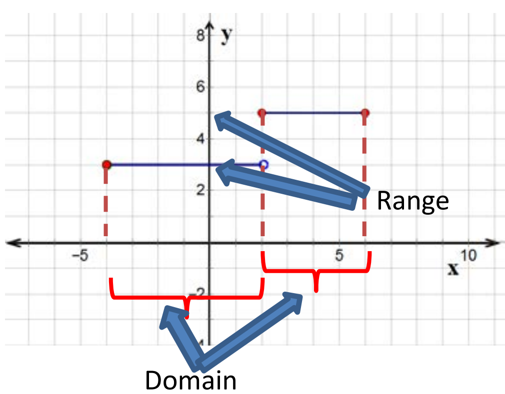
Domain:$[-6,2]$

Range: $\{\mathbf{5},\mathbf{1}\}$

Each x coordinate between -3 and -1 has two y coordinates associated to it on the graph making y not a function of x.

o Function	o One-to-One
✓ Not a Function	o Not One-to-One

N.



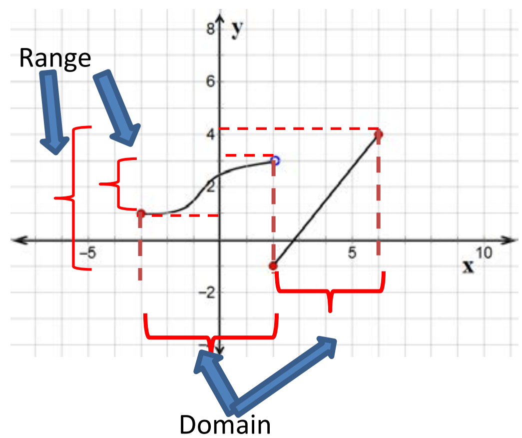
Each x coordinate has a unique y coordinate, so y is a function of x. But several x coordinates have the same output, so y is not a one-to-one function of x.

✓ Function	o One-to-One
o Not a Function	✓ Not One-to-One

Domain:$[\mathbf{-4},\mathbf{6}]$

Range: $\{\mathbf{2},\mathbf{5}\}$

O.



Each x coordinate has a unique y coordinate, so y is a function of x. But several x coordinates have the same output, so y is not a one-to-one function of x.

✓ Function	o One-to-One
o Not a Function	✓ Not One-to-One

Domain:$[\mathbf{-3},\mathbf{6}]$

Range: $[\mathbf{-1},\mathbf{4}]$

From above examples you can see that sometimes even when the domain is an interval, the range can be discrete points and other times it is an interval. So there is no fixed pattern, just pay careful attention to the function and remember the y-coordinates are the range values and the x-coordinates are the domain values.

Cover Sheet for Sec 1.1   Name: ______________________________

Summary

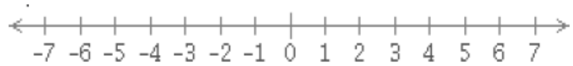
Concept	In your words describe what these concepts mean to you
1. **Relation**	
2. **Function**	
3. **Domain of a function**	
4. **Range of a function**	
5. **Different ways to represent a function (give examples you make up not the ones appear in the book)**	
6. **One-to-one function**	

Please list any problems you were stuck on or concepts that you would like more help on in class.

Video Log 1.1

1. Graph the set $\{x \mid -2 \le x < 0\}$ on the number line.

Then, write the set using interval notation.



2. The sets F and H are defined as follows.

$$F = \{x \mid x > 1\}$$

$$H = \{x \mid x \le 6\}$$

Write $F \cup H$ and $F \cap H$ using interval notation.
If the set is empty, write $\emptyset$

The sets D and E are defined as follows.

$$D = \{z \mid z \ge 4\}$$

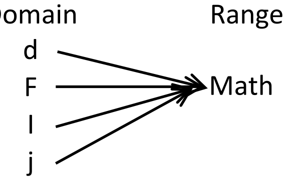
$$E = \{z \mid z > 9\}$$

Write $D \cup E$ and $D \cap E$ using interval notation.
If the set is empty, write $\emptyset$

3. Determine the following
   - i. Is the relation a function or not.
   - ii. Is the relation a one-to-one function
   - iii. Domain and Range

A.

Domain Range



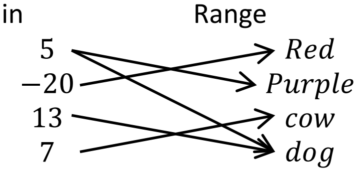
o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

B.

Domain Range



o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

C.

$$\{(-1,b),(2,8),(5,-3),(2,-3)\}$$

o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

D.

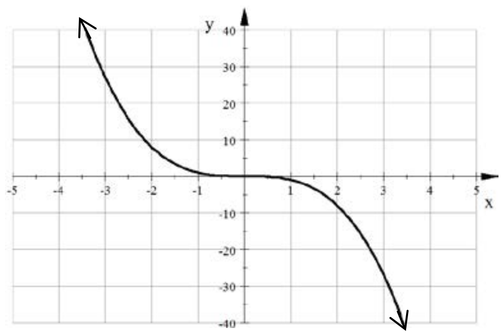
$$\{(2,-1),(2,1),(c,2),(d,4)\}$$

o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

E.

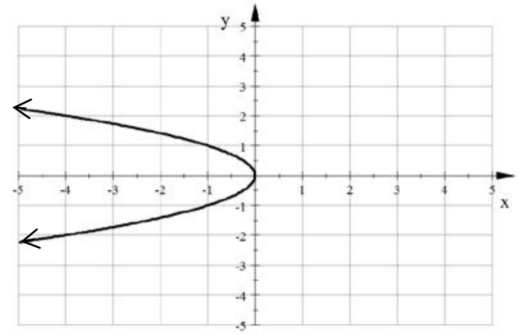


o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

F.



o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

G. $f(x)=3\sqrt{x-1}$

o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

H. $x^2+y^2=9$

o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

I. $Temp(t) = \frac{9}{5}t + 32$

o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

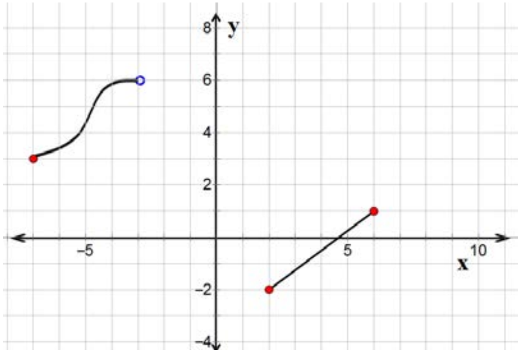
J. $y^5 = x$

o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

K.

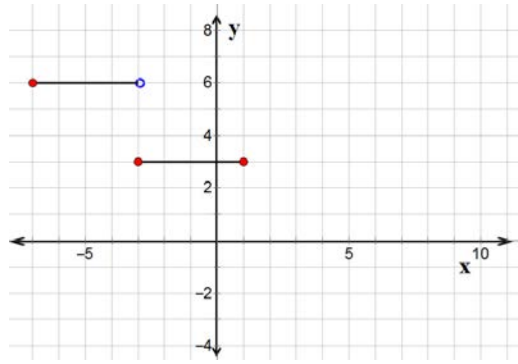


o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

L.

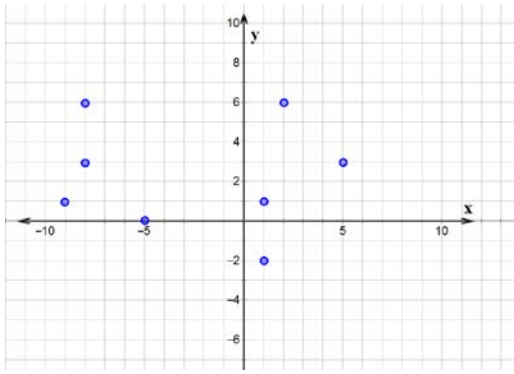


o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

M.

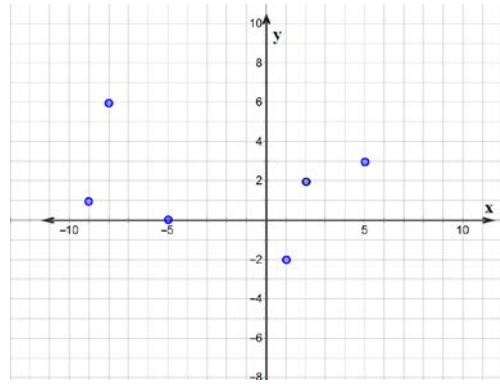


o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

N.



o Function	o One-to-One
o Not a Function	o Not One-to-One

Domain:

Range:

4. For each if the following equations, determine whether

y is a function of x	x is a function of y
A. $y^2 = 2x$ ☐ Function ☐ Not a function	B. $7x - 5y = 10$ ☐ Function ☐ Not a function
C. $2x - 7y = 9$ ☐ Function ☐ Not a function	D. $x = 3y^2$ ☐ Function ☐ Not a function
E. $6x + \|y\| = 3$ ☐ Function ☐ Not a function	F. $x^3 = y$ ☐ Function ☐ Not a function
G. $x^2 + 2 = y$ ☐ Function ☐ Not a function	H. $x^2y^2 = 9$ ☐ Function ☐ Not a function
I. $x^2 + 2y^2 = 8$ ☐ Function ☐ Not a function	J. $xy = 5$ ☐ Function ☐ Not a function

5. The function h is defined by the following rule: $h(x) = 4x + 5$. Complete the following table.

x	$h(x)$
-4	
-2	
2	
4	
5	

6. The functions f and g are defined as follows:

$f(x) = -3x + 2, \quad g(x) = 3x^3 + 5.$

Find $f(3)$ and $g(-3)$. Simplify your answers as much as possible.

$f(3) =$

$g(3) =$

7. The function f is defined as follows: $f(x) = \frac{4x}{3x-15}$. Find $f(4)$. Simplify your answer as much as possible.

8. The function h is defined as follows: $h(x) = \frac{x^2-3x-10}{x^2-14x+45}$. Find $h(6)$. Simplify your answer as much as possible.

9. Fill the table using the function rule $f(x) = \sqrt{x} + 6$. Simplify your answers as much as possible.

x	$f(x)$
-9	
0	
$\frac{1}{4}$	
4	
25	

10. The function h is defined as follows: $h(x) = \sqrt[3]{x-1}$.

Find $h(9), h(126), h(-26)$. Simplify your answer as much as possible.

1.2 Types of Functions

- Functions and Relations Part 2 (10:21 min) https://www.youtube.com/watch?v=RZwGOfAeqp4
- Functions and Relations Part 3 (10:41 min)https://www.youtube.com/watch?v=gPBE_QqVwbk

Now that you know how to work with relations to determine if they are functions and even if they are one-to-one functions is to play and create a lot of exciting new functions. Let our imagination go crazy and see what we can come up with. This is a chance for you to have some fun. We will show you some ways you can play with mathematical objects we called functions and then see where it takes us. Remember in the previous section we talked about one-to-one functions.

One-to-One Function: A function in which each output has one and only one input is called a <u>one-to-one</u> function.

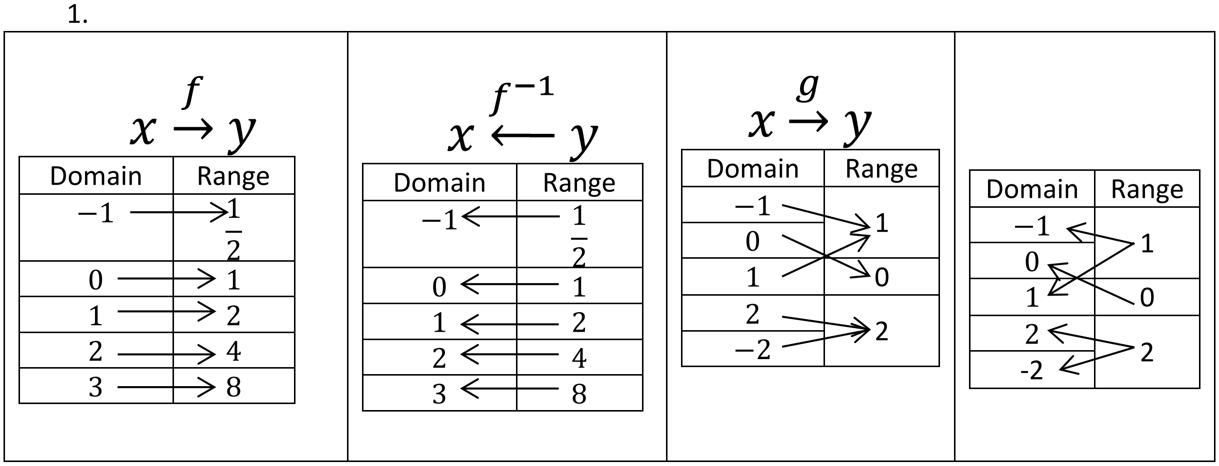
Looking this definition we can see that the first property of the inverse functions we can play with is that by reversing the roles of domain and range, domain becomes range and range becomes domain that this new relation will be a function.

Inverse Function: If $y = f(x)$ is a one-to-one function, then the function denoted by f^{-1} is called the inverse function if and only if for every x in the domain of f and for every y in the range of f they are mapped in reverse order, i.e., $x = f^{-1}(y)$. In other words the inverse function simply maps y to x.

Note: The notation f^{-1} is the name of the inverse function and the -1 is not an exponent.

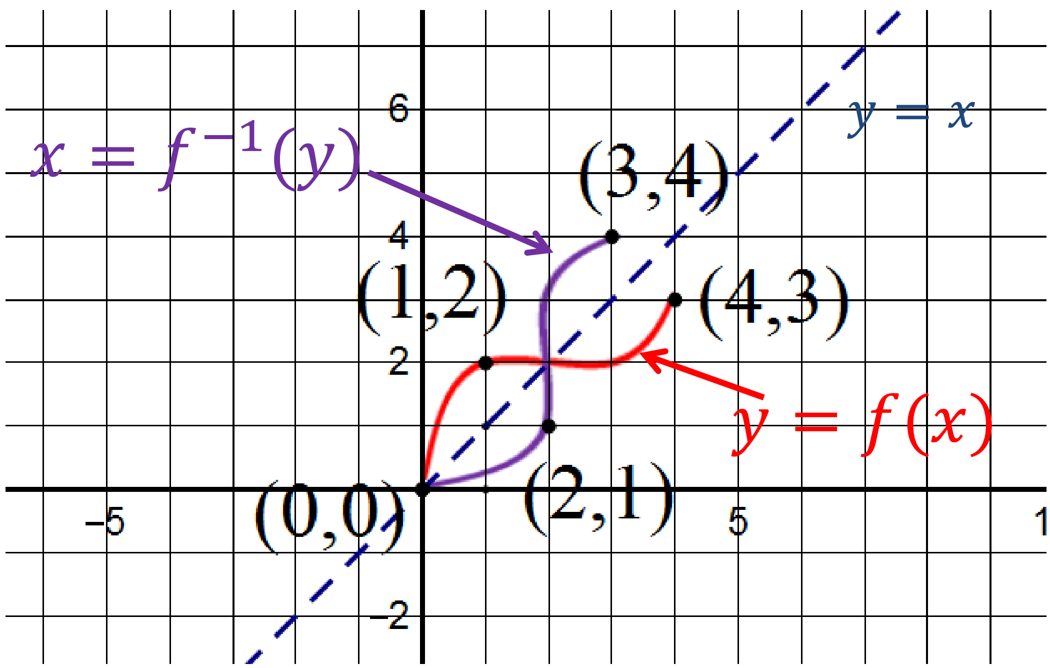
For example take a look at the functions below

1.



As you can see the function f is a one-to-one function. You can also see that if we reverse the relation it remains a function. This is the concept of the inverse function. You can see in the inverse function the roles that x and y play are reversed. Domain of $f = \{-1,0,1,2,3\}$ = Range of f^{-1} Range of $f = \left\{\frac{1}{2},1,2,4,8\right\}$ = Domain of f^{-1}	As you can see the relation g is not a function. We can still look at the inverse relation but we won't have a function to work with. So we only really will concentrate on the one-to-one functions when working with the concept of inverse functions.

2. If $f: \{(1,a),(-2,t),(4,g),(8,p)\}$ (which is a one-to-one function), then the function where the domain values become the range values and vice versa for each of the pairs is the inverse function. So $f^{-1}: \{(a,1),(t,-2),(g,4),(p,8)\}$ this new relation is also a function.
   Domain of $f = \{1,-2,4,8\}$ = Range of f^{-1}
   Range of $f = \{a,t,g,p\}$ = Domain of f^{-1}
3. If $y = f(x)$ is the graph shown below in red, then the purple graph would represent its inverse function. Since all points (x,y) on the original graph would become (y,x) in the inverse function graph, you can see that all points where $y = x$ will remain the same. So essentially it means if we graph the function $y = f(x)$ its inverse function is the same graph reflected across the line $y = x$.



Steps to graph inverse functions: To sketch the graph of $y = f^{-1}(x)$, switch the $x-, y$-role of all the points on the graph of $y = f(x)$. If $(x = a,\ y = f(a) = b)$, or $(a,\ b)$, is on the graph of f, then $(x = b, y = a)$ is on the graph of $y = f^{-1}(x)$.

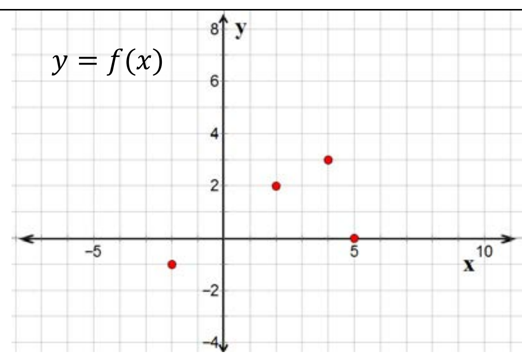
Steps to finding inverse functions:
Step 1: Write the original one-to-one function as $y = f(x)$
Step 2: Switch the roles of x and y so we have $x = f(y)$
Step 3: Solving for y will give us the inverse function $f^{-1}(x)$.

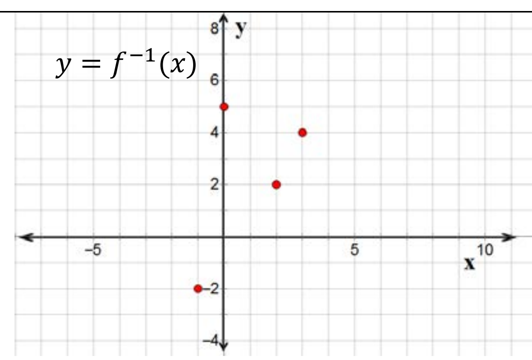
Examples

Find the inverses of the functions below.

1. $f(x) = 2x - 3$
   Step 1: $y = 2x - 3$
   Step 2:$x = 2y - 3$
   Step 3: $x + 3 = 2y$
   $\frac{x+3}{2} = y$
   Solution: $f^{-1}(x) = \frac{x+3}{2}$
   Since both the function and its inverses are lines
   Domain of f: $(-\infty, \infty)$ = Range of f^{-1}
   Range of f: $(-\infty, \infty)$ = Domain of f^{-1}

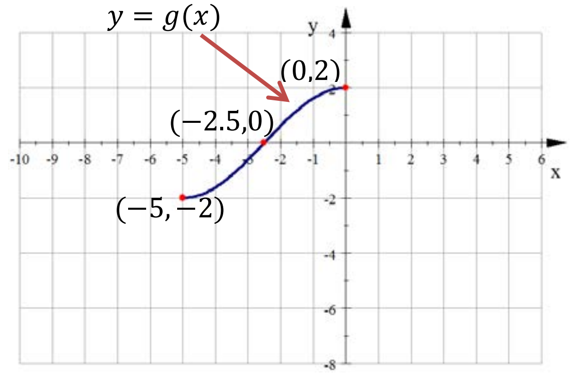
2. $f(x) = x^2, for\ x \geq 0$
   Step 1: $y = x^2\ for\ x \geq 0$ this restriction of domain is necessary to make the function one-to-one.
   Step 2:$x = y^2\ for\ y \geq 0$ which would mean $x \geq 0$
   Step 3: $\sqrt{x} = y\ for\ x \geq 0$since
   Solution: $f^{-1}(x) = \sqrt{x}, x \geq 0$
   Domain of f: $[0, \infty)$ = Range of f^{-1}
   Range of f: $[0, \infty)$ = Domain of f^{-1}
3. Sketch the graph of the inverse functions shown below. Sketch the graph of the inverse function. Find the domain and range of the original function and its inverse.

<table>
<tr><td colspan="2">a. $y = f(x)$.</td></tr>
<tr><td>


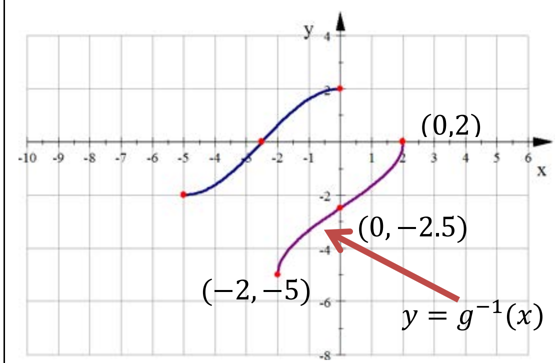
Domain of $f = \{-2,2,4,5\}$
Range of $f = \{-1,2,3,0\}$
</td><td>


Domain of $f = \{-2,2,4,5\}$
Range of $f = \{-1,2,3,0\}$
</td></tr>
</table>

b. $y = g(x)$



Domain of $g = [-5,0]$
Range of $g = [-2,2]$

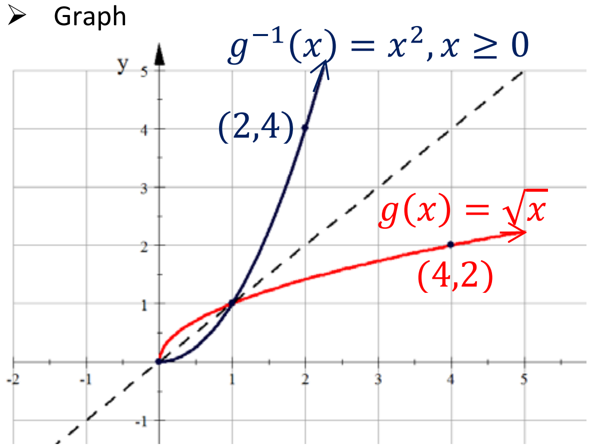


Domain $g^{-1} = [-2,2]$
Range of $g^{-1} = [-5,0]$

Other Types of Functions

1. Square Root Function: A function defined as $g(x) = \sqrt{x}$, for all real numbers $x \geq 0$.

- Graph



- One-to-One: Yes
- Inverse function $g^{-1}(x) = x^2, for\ x \geq 0$

Step 1: $y = \sqrt{x}, for\ x \geq 0$
Step 2: $x = \sqrt{y}, for\ y \geq 0$
Step 3: $x^2 = y\ for\ y \geq 0$, and therefore $x \geq 0$

- Domain of g: $[0, \infty)$ = Range of g^{-1}
- Range of g:$[0, \infty)$ = Domain of g^{-1}

Examples: For $g(x) = \sqrt{x}$ find the values of

a. $g\left(\frac{2}{3}\right)$
b. $g(100)$
c. $g(3456)$

Solution

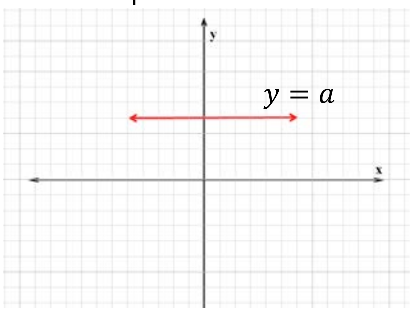
a. $g\left(\frac{2}{3}\right) = \sqrt{\frac{2}{3}}$
b. $g(100) = \sqrt{100} = 10$
c. $g(3456) = \sqrt{3456}$

2. **Polynomial function:** A function defined as $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n$, where $a_0,\ a_1,\ a_2,\ a_{3,}\ldots a_n$ are all real numbers and n≥ 0 is a whole number. Domain and range of these functions is all real numbers.

Example

i) **Constant Function:** A function defined as $f(x) = a$, where a is any real number.
A special case of the polynomial function of degree zero.

- Generic Graph

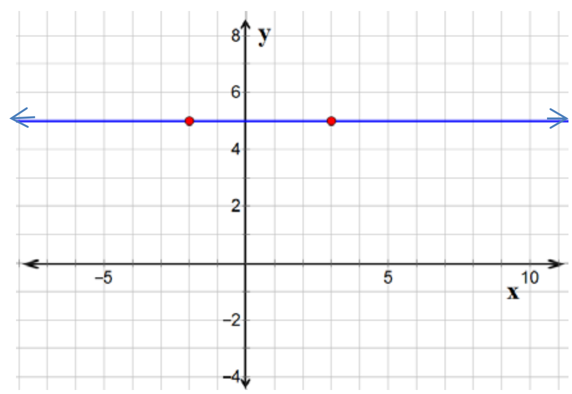


- Domain: All real numbers
- Range: $\{a\}$
- One-to-One: NO

Example

If $f(x) = 5$, then find

a. $f\left(\frac{2}{3}\right)$
b. $f(100)$
c. $f(-3456)$
d. $f(a+h)$
e. Sketch the graph



Solutions

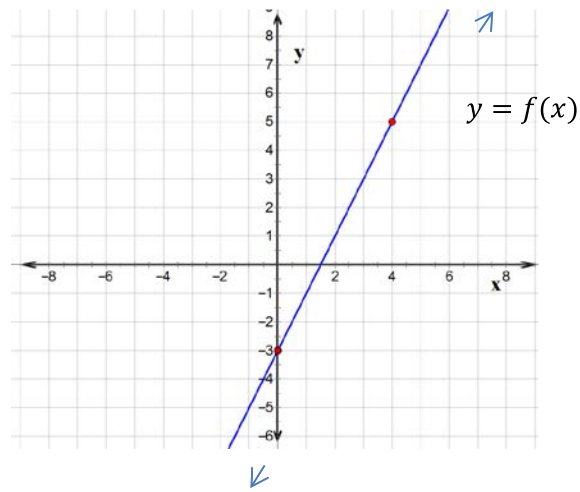
a. $f\left(\frac{2}{3}\right) = 5$
b. $f(100) = 5$
c. $f(-3456) = 5$
d. $f(a+h) = 5$
e. As you can see from the parts a,b,and c no matter what x-coordinate you plot the y-coordinate is always 5 so it is a horizontal line as shown above.

ii) Linear Function: A function defined as $(x) = mx + b$.
A special case of polynomial function with degree one. Remember m =slope of the line, $b = y$-intercept.

Example

1. If $f(x) = 2x - 3$, then

a. Find $f(-1)$
b. Find $f(0)$
c. Find $f(4)$
d. Find the inverse function
e. Sketch the graph of $y = f(x)$



Solution

a. $f(-1) = 2(-1) - 3 = -2 - 3 = -5$
b. $f(0) = -3$
c. $f(4) = 2(4) - 3 = 8 - 3 = 5$
d. Inverse function

$$y = 2x - 3$$
$$x = 2y - 3$$
$$x + 3 = 2y$$
$$\frac{x+3}{2} = y$$

Inverse function is $f^{-1}(x) = \frac{x+3}{2}$

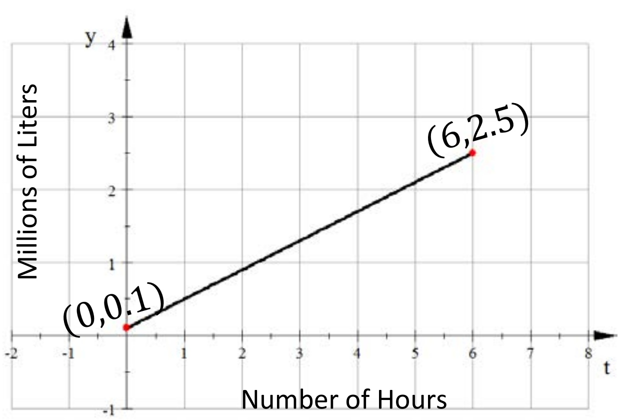
e. Graph of the function is to the right with slope of 2 and y-intercept of -3.

2. An Olympic size swimming pool holds 2,500,000 liters of water. If the pool currently holds 100,000 liters of water, and water is being pumped at 400,000 liters/hour into the tank.

c. Write a function that represents the amount of water in the tank at t hours.
d. Find the domain and range of this function.
e. Sketch the graph of this function.

Solution

a. $A(t) = 100000 + 400000t$ Liters, for $0 \leq t \leq 6$
The reason for the restriction is time is never negative and after 6 hours the tank will be full since it holds a maximum of 2,500,000 liters of water.
b. Domain of $A = [0,6]$ and Range of $A = [100000,2500000]$
c. Graph is to the right.
The scale is each tick mark represents 1 million liters on the y-axis and 1 hour on the t-axis.



iii) Square Function: A function defined as $(x) = ax^2 + bx + c$.
A special case of polynomial function with degree two

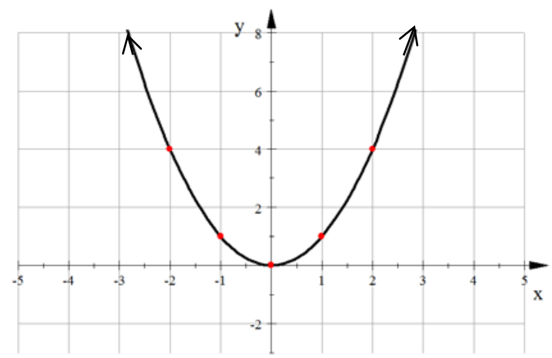
Example If $f(x) = x^2$, then

a. Find $f\left(\frac{2}{3}\right)$
b. Find $f(-2)$
c. Find $f(2)$
d. Find $f(a+h)$
e. Sketch the graph the function $y = x^2$
f. Is the function one-to-one?

Solution

a. $f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$
b. $f(-2) = (-2)^2 = 4$
c. $f(2) = (2)^2 = 4$
d. $f(a+h) = (a+h)^2 = a^2 + 2ah + h^2$
e. Plot a few points to sketch the graph of the function. See to the left
f. No. The function is not one-to-one as it does not pass the horizontal line test.

x	$y = x^2$
-2	$(-2)^2 = 4$
-1	$(-1)^2 = 1$
0	$(0)^2 = 0$
1	$(1)^2 = 1$
2	$(2)^2 = 4$



iv) Cube Function: A function defined as $f(x) = a + bx + cx^2 + dx^3$
A special case of polynomial function with degree three

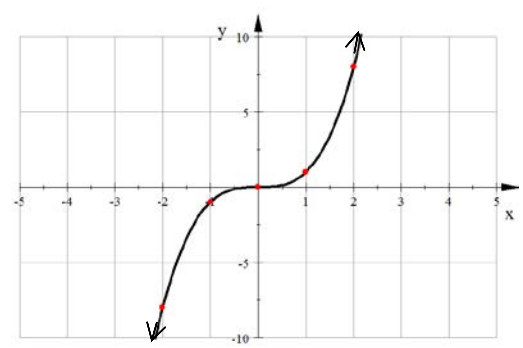
Example If $f(x) = x^3$, then

a. Find $f\left(\frac{2}{3}\right)$
b. Find $f(-2)$
c. Find $f(2)$
d. Sketch the graph the function $y = x^3$
e. Is the function one-to-one?

Solution

a. $f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$
b. $f(-2) = (-2)^3 = -8$
c. $f(2) = (2)^3 = 8$
d. Plot a few points to sketch the graph of the function. See to the left
e. Yes. The function is one-to-one as it does pass the horizontal line test.
f. Finding inverse function write $y = x^3$, then we get $x = y^3$ solving for y we get that inverse function is $f^{-1}(x) = \sqrt[3]{x}$

x	$y = x^3$
-2	$(-2)^3 = -8$
-1	$(-1)^3 = -1$
0	$(0)^3 = 0$
1	$(1)^3 = 1$
2	$(2)^3 = 8$



4. Piecewise Defined Function: A piecewise defined function is exactly what its name suggests. It is defined by two or more equations for different parts of the domain.

Examples:

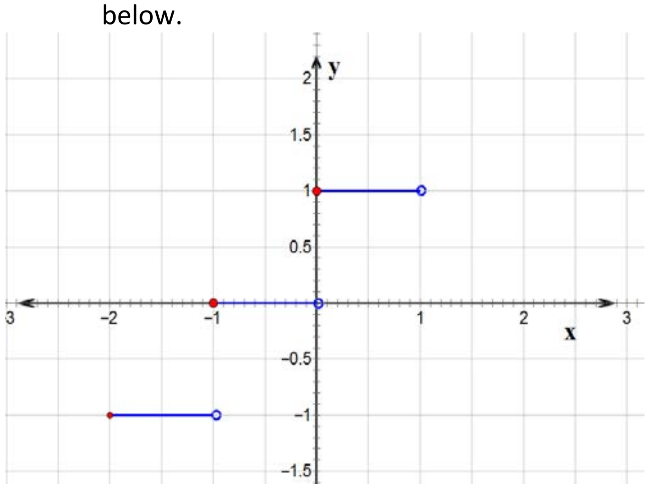
I. $$f(x) = \begin{cases} 1, & for \quad 0 \le x < 1 \\ 0, & for \ -1 \le x < 0 \\ -1, & for \ -2 \le x < -1 \end{cases}$$

Evaluate:

a) Graph the function f
b) Domain of f
c) Range of f
d) $f(-0.5)$
e) $f(0.5)$
f) $f(-1.5)$

Solution

a) Our function can be thought of as being made up of three separate functions. So to plot the graph we have plot each of the three separate functions in one graph. You can do so by plotting some points until you know how it looks like. In this case all three pieces are constant functions in the given interval so the graph would be as shown below.



Range

Domain

$$f(x) = \begin{cases} \boxed{1,} & for \quad 0 \le x < 1 \\ \boxed{0,} & for \ -1 \le x < 0 \\ \boxed{-1,} & for \ -2 \le x < -1 \end{cases}$$

b) Domain of $f = [-2, 1)$
c) Range of $f = \{-1,0,1\}$
d) $f(-0.5) = 0$ since $x = -0.5$ falls in the interval $-1 \le x < 0$
e) $f(0.5) = 1$ since $x = 0.5$ falls in the interval $0 \le x < 1$
f) $f(-1.5) = -1$ since $x = -1.5$ falls in the interval $-2 \le x < -1$

II. $f(x)=\begin{cases}-\frac{1}{3}x-1, & for \quad -3\le x<0\\ -2, & for \quad 0\le x<2\\ 2x-4, & for \quad 3\le x\end{cases}$

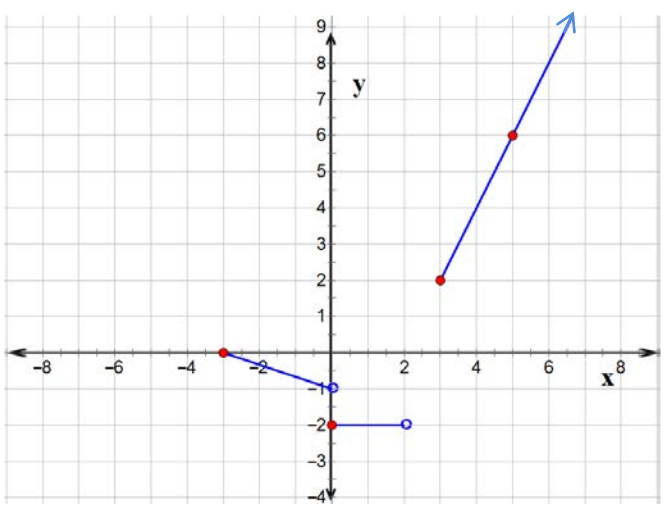
Evaluate:
a) Domain of f
b) Range of f
c) Graph the function f
d) $f(-3)$
e) $f\left(-\frac{1}{2}\right)$
f) $f(1.5)$
g) $f(5)$
h)

Solution

c) Our function can be thought of as being made up of three separate functions. So to plot the graph we have plot each of the three separate functions in one graph. The first and the last piece are lines so plotting the two end points of the line segments it is made up of will suffice, and the middle is a constant function like we have seen before.

x	$y=-\frac{1}{3}x-1$
-3	$-\frac{1}{3}(-3)-1$ $=1-1=0$
0	$-\frac{1}{3}(0)-1$ $=0-1=-1$

x	$y=2x-4$
3	$2(3)-4$ $=6-4=2$
5	$2(5)-4$ $=10-4=6$



Solution

Range

Domain

$$f(x)=\begin{cases}\boxed{-\frac{1}{3}x-1,} & for \quad -3\le x<0\\ \boxed{-2,} & for \quad 0\le x<2\\ \boxed{2x-4,} & for \quad 3\le x\end{cases}$$

a) Domain of $f=[-3,0)\cup[3,\infty)$
Look at the graph to see what x-cordinates are involved in the graph.

b) Range of $f=\{-2\}\cup(-1,0]\cup[2,\infty)$
Look at the graph to see what y-cordinates are involved in the graph.

d) $f(-3)=-\frac{1}{3}(-3)-1=1-1=0$
since $x=-3$ falls in the interval $-3\le x<0$

e) $f\left(-\frac{1}{2}\right)-\frac{1}{3}\left(-\frac{1}{2}\right)-1=\frac{1}{6}-1=-\frac{5}{6}$
since $x=-\frac{1}{2}$ falls in the interval $3\le x<0$

f) $f(1.5)=-2$ since $x=-1.5$ falls in the interval $0\le x<2$

g) $f(5)=2(5)-4=10-4=6$

Before we get into the next few types of functions we need to get familiar with a few definitions.

5. **Rational Function:** A rational function is defined as the rational of two polynomial functions denoted as $R(x) = \frac{P(x)}{Q(x)}$, for polynomial function $P(x)$, $Q(x)$ and valid for real numbers x for which $Q(x) \neq 0$.
Domain of $R(x) =$ all real numbers where $Q(x) \neq 0$

Examples

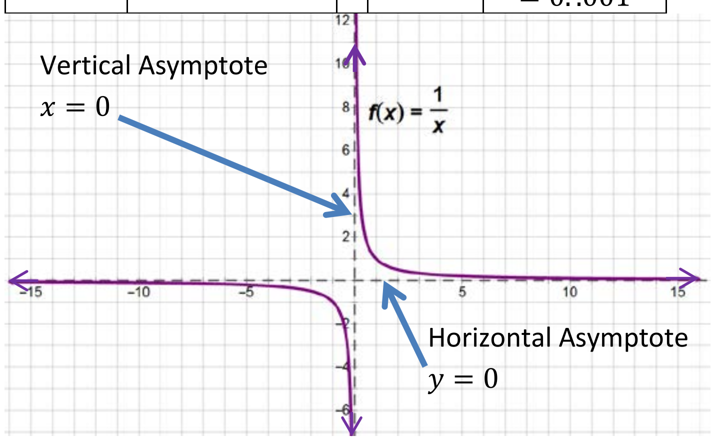
I. $R(x) = \frac{1}{x}$
   a. Find $R(2)$
   b. Domain of $R(x)$
   c. Sketch the graph of $R(x)$
   d. Is $R(x)$ one-to-one?
   e. What is the inverse function of $R(x)$

Solution

a. $R(2) = \frac{1}{2}$
b. Domain of $R(x) =$ all real numbers $x \neq 0$
   $= (-\infty, 0) \cup (0, \infty)$
c. To sketch the graph of a function we do not know much about for now involves plotting points. As we can see the denominator cannot be zero. See the graph to the right. As you can see this graph is unique in that as x approaches zero from the positive side the y coordinate shoots to infinity, and x approaches zero from the negative side the y coordinate shoots to negative infinity. Also when x shoots to infinity, the y coordinate approaches zero from the positive side and x shoots to negative infinity, the y coordinate approaches zero from the negative side. The lines $x = 0$ and $y = 0$ are called horizontal and vertical asymptotes respectively. We will study these in-depth later.
d. Yes. $R(x)$ passes the horizontal line test.

To sketch the graph of a function we do not know much about for now involves plotting points. As we can see the denominator cannot be zero.

x	$y = \frac{1}{x}$		x	$y = \frac{1}{x}$
-0.1	$\frac{1}{-0.1} = -10$		-10	$\frac{1}{-10} = -0.1$
-0.01	$\frac{1}{-0.01} = -100$		-100	$\frac{1}{-100} = -0.01$
-0.001	$\frac{1}{-0.001} = -1000$		-1000	$\frac{1}{-1000} = -0..001$
0.1	$\frac{1}{0.1} = 10$		10	$\frac{1}{10} = 0.1$
0.01	$\frac{1}{0.01} = 100$		100	$\frac{1}{100} = 0.01$
0.001	$\frac{1}{0.001} = 1000$		1000	$\frac{1}{1000} = 0..001$



From the chart the closer to zero the x values come from left or the right of zero, the y values either shoot to ∞, or $-\infty$ respectively. In other words, if we were to zoom in closer and closer to $x = 0$, the graph resembles the vertical line $x = 0$. Such a line is then denoted by the dotted line you see and is called a horizontal asymptote. Similarly if we zoom out so that x values shoot to either ∞ or $-\infty$ then the y coordinate seems to get closer to zero from either above or below respectively. In other words, we were to zoom out a lot, the graph would resemble the line $y = 0$. Such a line is called the horizontal asymptote.

e. Inverse function

$y = \frac{1}{x}$

$x = \frac{1}{y}$ solving for y we get

$y = \frac{1}{x}$ or $R^{-1}(x) = \frac{1}{x}$

We will look at inverse functions of rational functions a little later in more depth.

II. $R(x) = \frac{2x-1}{x^2-4x+3} = \frac{2x-1}{(x-3)(x-1)}$

a. Find $R(5)$

b. Domain of $R(x)$

Solution

a. $R(5) = \frac{2(5)-1}{(5-3)(5-1)} = \frac{10-1}{(2)(4)} = \frac{9}{8}$

b. Domain of $R(x)$ = All real numbers for which denominator is non-zero or $(x-3)(x-1) \neq 0$

= All real numbers so that $x \neq 3$ or $x \neq 1$

$= (-\infty, 1) \cup (1,3) \cup (3, \infty)$

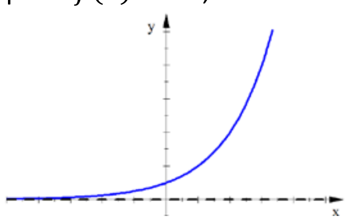
6. **Exponential Function:** An exponential function is defined as $f(x) = a^x$, where a is a positive real number not equal to 1.

Properties of an exponential function

- Domain: All real numbers
- Range: $(0, \infty)$
- One-to-One: Yes!
- Points on the graph to plot are

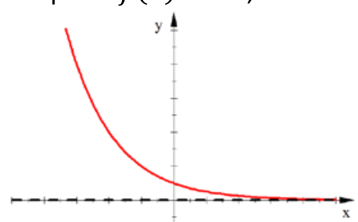
x	$y = a^x$
-2	$a^{-2} = \frac{1}{a^2}$
-1	$a^{-1} = \frac{1}{a}$
0	$a^0 = 1$
1	$a^1 = a$
2	a^2

Graph of $f(x) = a^x$, for $a > 1$



Horizontal Asymptote $y = 0$

Graph of $f(x) = a^x$, for $a < 1$



Horizontal Asymptote $y = 0$

For exponential functions we can see that for all $a > 0$ the negative powers of a gives us values of the type $\frac{1}{a^n}$ where $n > 0$. The value of $\frac{1}{a^n}$ will keep on getting smaller and smaller taking the y-coordinate to zero either as x goes to ∞, or $-\infty$ depending on whether $a > 1$ or $0 < a < 1$ respectively. Therefore the line $y = 0$ is called the horizontal asymptote. In other words, the graph resembles the line $y = 0$ when we zoom out sufficiently.

Example

I. If $f(x) = 2^x$, then

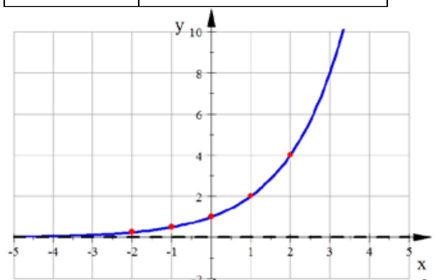
a. Find $f(-1)$
b. Find $f(3)$
c. Find $f(-3)$
d. Find $f(a+h)$
e. Sketch the graph of the function.

Solution

a. $f(-1) = (2)^{-1} = \frac{1}{2}$
b. $f(3) = 2^3 = 8$
c. $f(-3) = 2^{-3} = \frac{1}{8}$
d. $f(a+h) = 2^{a+h}$
e. Sketch the graph of the function.
See graph to the right. You can see from the chart that as x approaches $-\infty$ the y-coordinate approaches zero. This makes $y = 0$ the horizontal asymptote.

Graph

x	$Y = 2^x$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$



Horizontal Asymptote $y = 0$

Example

II. If $f(x) = \left(\frac{1}{3}\right)^x$, then

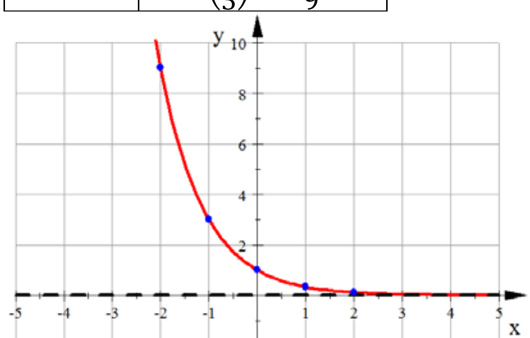
a. Find $f(-1)$
b. Find $f(2)$
c. Find $f(-2)$
d. Sketch the graph of the function.

Solution

a. $f(-1) = \left(\frac{1}{3}\right)^{-1} = 3$
b. $f(2) = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$
c. $f(-2) = \left(\frac{1}{3}\right)^{-2} = 3^2 = 9$
d. Sketch the graph of the function.
See graph to the right. You can see from the chart that as x approaches ∞ the y-coordinate approaches zero. This makes $y = 0$ the horizontal asymptote.

Graph

x	$y = \left(\frac{1}{3}\right)^x$
-2	$\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$
-1	$\left(\frac{1}{3}\right)^{-1} = 3$
0	$\left(\frac{1}{3}\right)^0 = 1$
1	$\left(\frac{1}{3}\right)^1 = \frac{1}{3}$
2	$\left(\frac{1}{3}\right)^2 = \frac{1}{9}$



Horizontal Asymptote $y = 0$

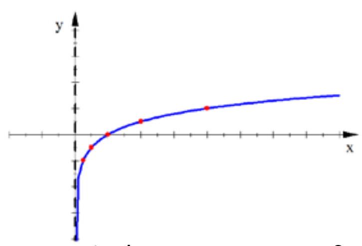
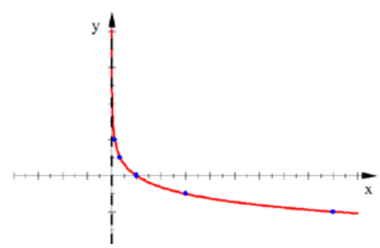
Playing

As you can see all the exponential functions discussed here are one-to-one and therefore have inverses. If we follow our steps to find an inverse function of say $f(x) = 2^x$ we get the following $y = 2^x$ and then $x = 2^y$. At this step we now solve the equation for y but we cannot since y is in the exponent so mathematicians created a new function to represent this inverse function called logarithmic function. For the inverse function of $f(x) = 2^x$ we will represent the inverse function as $f^{-1}(x) = log_2(x)$ read as log base 2 of x. What does it mean to evaluate values of this inverse function?

First note that $x = 2^y$ and $y = log_2(x)$ are equivalent equations. So if we wanted to find the value of $y = log_2(8)$ then we need to find exponent y for which $8 = 2^y$. Which means $y = 3$. Similarly you can play and see that $log_2\left(\frac{1}{2}\right) = -1$ since $\frac{1}{2} = 2^{-1}$.

Note that since $y = 0$ is the horizontal asymptote to the function $y = 2^x$, the line $x = 0$ will be the vertical asymptote to the function $y = log_2(x)$ due to properties of inverse functions.

7. **Logarithmic Function:** A function defined as $f(x) = log_a x$, where $a > 0$, $a \neq 1$ and is positive real number.

Properties of the logarithmic function:	Graph of $f(x) = log_a x$, for $a > 1$	Graph of $f(x) = log_a x$, for $0 < a < 1$
➢ To compute the value of $f(x)$ means answering: $a^? = x$ ➢ Graph: ➢ Domain: $(0, \infty)$ ➢ Range: All real numbers ➢ One-to-One: Yes! ➢ Inverse function $f^{-1}(x) = a^x$	 Vertical Asymptote $x = 0$	 Vertical Asymptote $x = 0$

Cover Sheet for Sec 1.2 Name: ________________________________

Summary

Concept	In your words describe what these concepts mean to you
One-to-One function	
Inverse function	
Relationship of domain and range of inverse functions to the original function	
Polynomial functions	
Rational functions	
Exponential functions	
Logarithmic functions	

Video log 1.2

1. The function f is defined by the following rule: $f(x) = 8^x$. Find $f(x)$ for each x-value in the table.

x	$f(x) = 8^x$
-2	
-1	
0	
1	
2	

2. The tables below give exponential functions. Write the equation for each of the functions.

x	$f(x)$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4

x	$g(x)$
-2	$\frac{1}{e^2}$
-1	$\frac{1}{e}$
0	1
1	e
2	e^2

$f(x) =$____________ $g(x) =$____________

3. The functions f is defined as follows:

$$f(x) = \begin{cases} \frac{3}{4}x + 2 & if\ x \neq -1 \\ 2 & if\ \ x = -1 \end{cases}$$

Find the following.

$f(-3)$

$f(-1)$

$f(-2)$

4. The function g is defined by $g(x) = \frac{3x-4}{x+5}$.
Find $g(x+5)$.

5. $f(x) = \sqrt{x-2}$
   a. Find $f(6)$
   b. Find $f(2)$
   c. Find domain and range of the function.
   d. Is f one-to-one?
   e. If you answered yes to part d, find the inverse function.

6. Find the domain and range of all the relations below either in interval notation or set notation as appropriate.

a) $f(x) = \frac{1}{x-4}$

Domain:

Range:

b) $g(x) = \sqrt{x-4}$

Domain:

Range:

c) $h(x) = \sqrt{4-x}$

Domain:

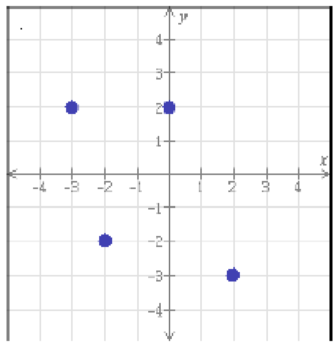
Range:

d) $r(x) = \sqrt{x-4} + 2$

Domain:

Range:

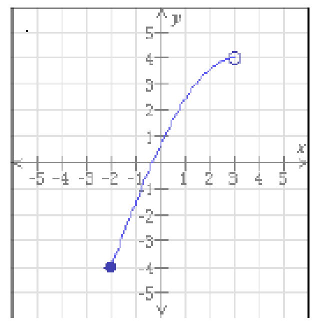
e)



Domain of T:

Range of T:

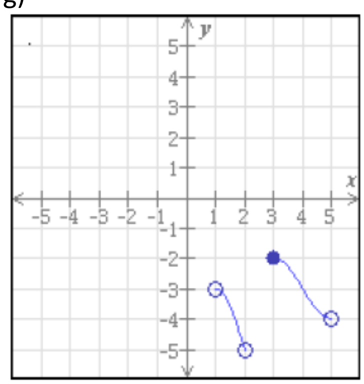
f)



Domain of h:

Range of h:

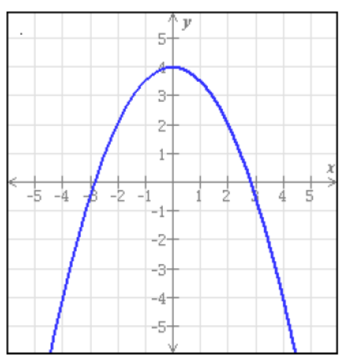
g)



Domain of S:

Range of S:

h)



Domain of f:

Range of f:

Evaluate $f(0) =$___________

Find the values of x, for which $f(x) = 0$

_____ , ______

i)

$$g(x) = \begin{cases} 2 & if\ x = 1 \\ -2 & if\ x < 1 \\ 3 & if\ x > 1 \end{cases}$$

Domain of g:

Range of g:

j)

$$p(x) = \begin{cases} 3\sqrt{x} & if\ x > 4 \\ 2x - 1 & if\ x < -4 \end{cases}$$

Domain of p:

Range of p:

7. A species of bacteria doubles every 30 minutes at room temperature. Write a function to represent the amount of these bacteria $A(t)$ at room temperature after t hours, if you initially started with 30000 bacteria.
   a. Find the number of bacteria after 2 hours.
   b. Find the number of bacteria at 4 hours.

Round your answer to the nearest whole number as necessary.

8. The radioactive substance Uranium-240 has a half-life of 14 hours. The amount $A(t)$ of a sample of Uranium-t240 remaining in grams after t hours is given by the exponential function $A(t) = 3000\left(\frac{1}{2}\right)^{\frac{t}{14}}$.
   a. Find the initial amount in the sample.
   b. Find the amount remaining after 30 hours.

   Round your answer to the nearest gram as necessary.

9. Suppose Rahul places $\$2000$ in an account that pays 5% interest compounded each year. Assume that no withdrawals are made from the account. Do not round your answers.
   a. Find the amount in the account at the end of 1 year.
   b. Find the amount in the account at the end of 2 years.

10. A car is purchased for $\$16{,}500$. After each year, the resale value decreases by 35%. What will the resale value be after 5 years? Round your answer to the nearest dollar.

11. Devon deposited $4000 into an account with 4.3% interest, compounded quarterly. Assuming the no withdrawals are made, how much will she have in the account after 10 years? Round your answer to the nearest dollar.

12. At the beginning of a population study, a city had 360,000 people. Each year since, the population has grown by 2.6%. Let t be the number of years since the start of the study. Let $P(t)$ represent the city's population at time t. Write an exponential function that represent the relationship between the population $P(t)$ and time t

13. Compare the function $f(x) = 3^x$ and $g(x) = 50x^2$ by completing parts a. and b.
    a. Fill in the table below. Note that the table is already filled in for $x = 4$

x	$f(x) = 3^x$	$g(x) = 50x^2$
4		
5		
6		
7		
8		

    b. For all x, $x \geq 5$ please check which of the following are true statements
    - ☐ $f(x) \geq g(x)$
    - ☐ $f(x) = g(x)$
    - ☐ $f(x) \leq g(x)$

    c. Explain your answer in part b.

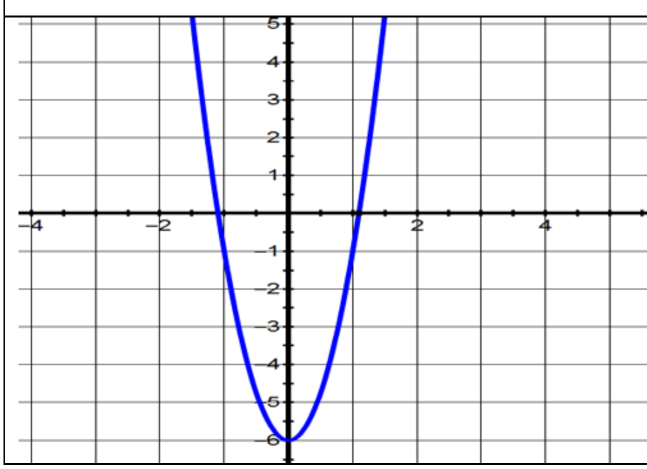
14. A corn factory delivers sugar to different stores. Let $C(S)$ represent the total cost to transport corn (in dollars). Let S be the amount of corn in tons transported. The company can transport up to 425 tons of corn. Suppose that $C(S) = 200S + 1500$ gives C as a function of S. Identify the correct description of the values in both the domain and range of the function. Then, for each choose the most appropriate set of values.

Domain	Range
Description of values ☐ Cost of transporting corn in dollars ☐ Amount of corn transported in tons Set of values ☐ The set of all real numbers from 0 to 425 ☐ The set of all real numbers from 1500 to 86500 ☐ The set of all real numbers from 200 to 1500 ☐ $\{425, 426, 427, \ldots\}$ ☐ $\{0, 1, 2, 3, \ldots 86500\}$ ☐ $\{200, 400, 600, 800, \ldots\}$	Description of values ☐ Cost of transporting corn in dollars ☐ Amount of corn transported in tons Set of values ☐ The set of all real numbers from 0 to 425 ☐ The set of all real numbers from 1500 to 86500 ☐ The set of all real numbers from 200 to 1500 ☐ $\{425, 426, 427, \ldots\}$ ☐ $\{0, 1, 2, 3, \ldots 86500\}$ ☐ $\{200, 400, 600, 800, \ldots\}$

15. A construction crew needs to pave a road that is 204 miles long. The crew paves 9 miles of the road each. The length, L (in miles), that remains to be paved after d days is given by the following function. $L(d) = 204 - 9d$.
Answer the following questions.
a. How many miles of the road the crew have left to pave after 13 days?
b. If 114 miles of the road remains to be paved, how many days has the crew been paving the road?

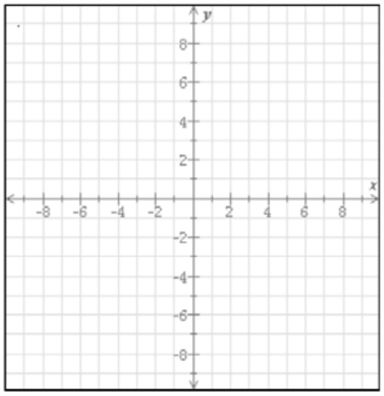
16.
Find the difference quotient $\frac{f(x+h)-f(x)}{h}$ where $h \neq 0$ for the function below. Explain what this quotient represents. Simplify your answer as much as possible.

$$f(x) = 5x^2 - 6$$



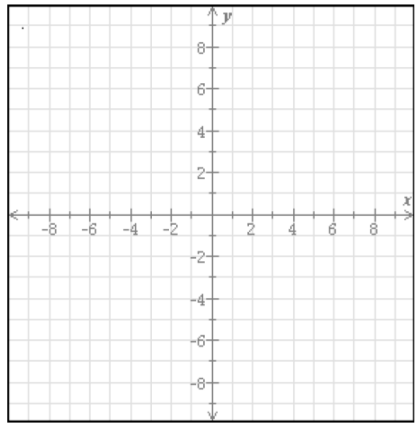
17. Sketch the graph of the functions and relations below. Explain clearly how you decided the graph was the shape you drew. Can you determine the domain and range of the functions and relations that you graphed based on the graphs.

a. $y = 2^x$



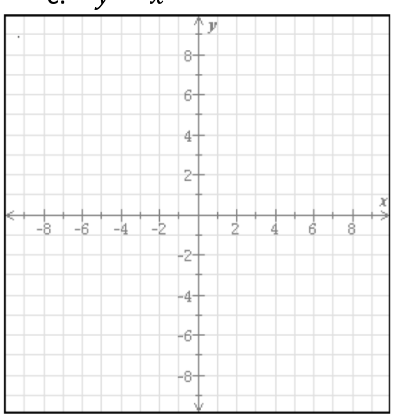
x	y

b. $y = \sqrt{x}$



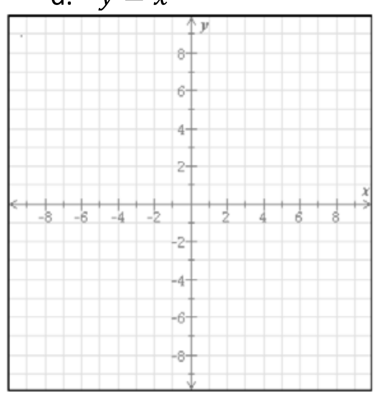
x	y

c. $y = x^3$



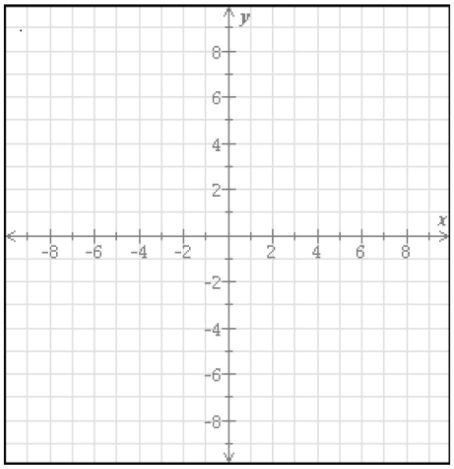
x	y

d. $y = x^4$



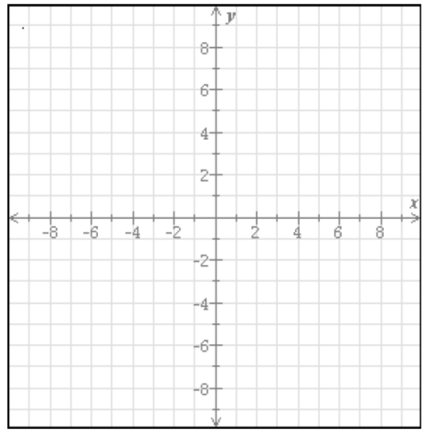
x	y

e. $y = 4^x$



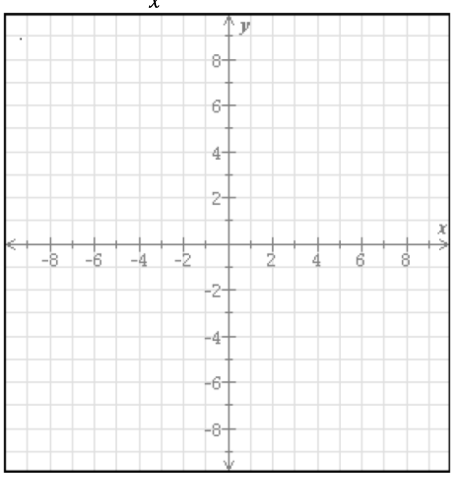
x	y

f. $y = \left(\frac{1}{4}\right)^x$



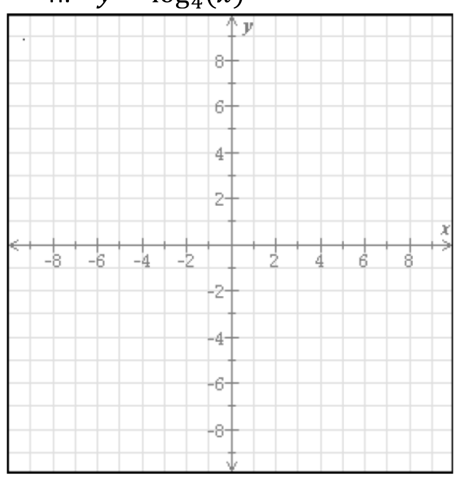
x	y

g. $y = \frac{3}{x}$



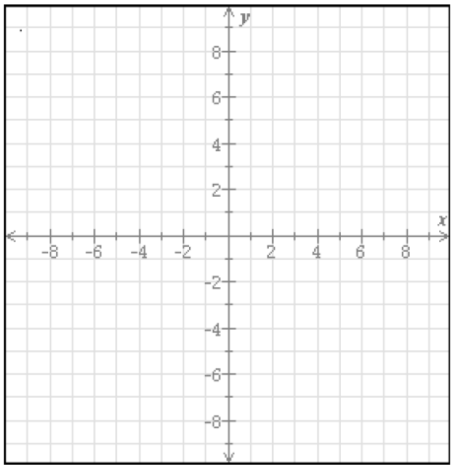
x	y

h. $y = \log_4(x)$



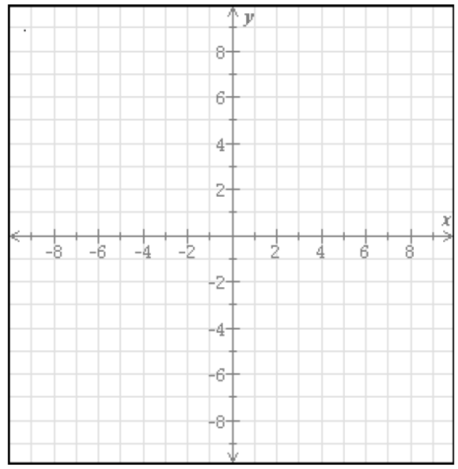
x	y

i. $y = \log_{10}(x)$



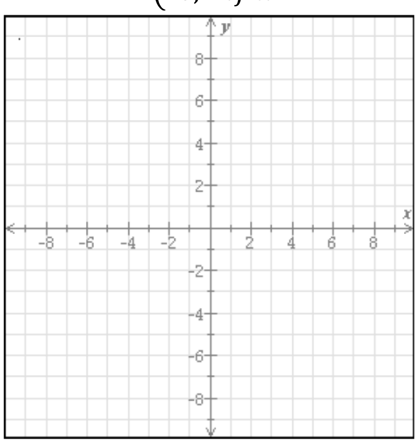
x	y

j. $y = log_3(x)$



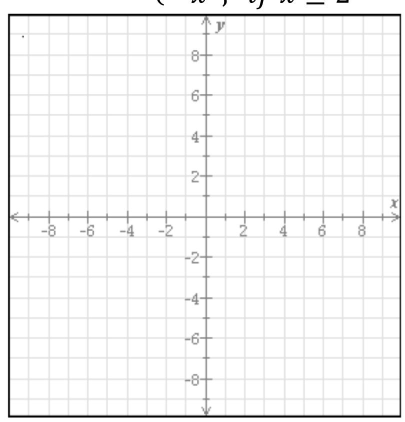
x	y

k. $y = \begin{cases} 2, & if\ x > 1 \\ -2, & if\ x < 1 \\ 0, & if\ x = 1 \end{cases}$

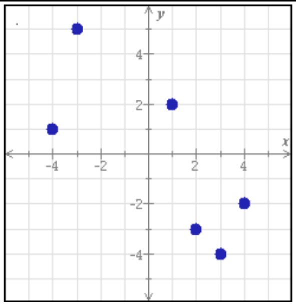
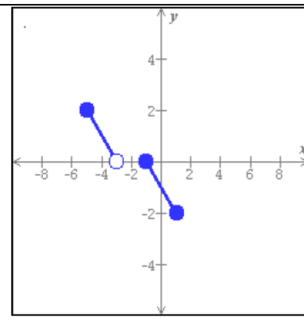


x	y

l. $y = \begin{cases} x+2, & if\ x > -2 \\ x^2, & if\ x \leq 2 \end{cases}$



x	y

18. Evaluate the following given the one-to-one functions below.			
	a. $f(1)$ b. $f^{-1}(1)$ c. $f(2)$ d. $f^{-1}(-4)$		a. $f(2)$ b. $f^{-1}(0)$ c. $f(-5)$ d. $f^{-1}(-2)$

19. Find the inverses of the following one-to-one functions. Then find the domains and ranges of the functions and their inverses.			
a. $f(x) = \frac{7x+1}{2x-1}$		b. $g(x) = \sqrt{2x-1}$ for $x \geq \frac{1}{2}$	
Domain of f	Range of f^{-1}	Domain of g	Range of g^{-1}
Domain of f^{-1}	Range of f	Domain of g^{-1}	Range of g
c. $h(x) = 4^x$		d. $p(x) = log_4 x$	
Domain of h	Range of h^{-1}	Domain of p	Range of p^{-1}
Domain of h^{-1}	Range of h	Domain of p^{-1}	Range of p